# Supplement to Two-Step Estimation of Network-Formation Models with Incomplete Information

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## A Extensions for Kernel Estimator

In this section, we maintain Assumptions 1, 3, and 4.

Consistency of the kernel estimator requires that the sequence of equilibria chosen by the equilibrium selection mechanism for every $n$ satisfies certain smoothness and finiteness conditions. This requires a modification of Assumption 2, redefining $\mathcal{G}(X^n, \theta_0)$ as the set of $s$-times differentiable, symmetric equilibria such that certain derivatives are uniformly bounded over $n$. We next provide conditions for the existence of such equilibria.

Let $\sigma : \mathbb{R}^m \to \mathbb{R}^p$. Define for $q \in \mathbb{N}^m$ $|q| = \sum_{i=1}^m q_i$ and the $|q|$th order derivative of $\sigma$ with respect to $x \in \mathbb{R}^m$ as $D_x^q \sigma(x) = \frac{\partial^{|q|}\sigma}{\partial x_1^{q_1} \dots \partial x_m^{q_m}}$. For the next proof, we will view $X$ as a vector and define $X_k$ to be its $k$th component. Additionally, for $i, j, k \in \mathcal{N}_n$, let $g_i \in A^n$ such that $g_{ii} = 0$, $g_{i,-jk}$ be $g_i$ with components $j$ and $k$ removed, and $(1, 1, g_{i,-jk})$ be $g_i$ with components $j$ and $k$ replaced with ones. Similarly define $g_{i,-j}$ and $(1, g_{i,-j})$.

**Theorem A.1.** *Suppose Assumption 5 holds and that $\mu$ and the joint CDF of $\varepsilon_i$ are $s$-times differentiable. Then there exists a sequence of equilibria $\{\sigma_n\}_{n=2}^\infty$ that are $s$-times differentiable at $X$ and symmetric such that for any $q$ with $|q| \leq s$,*

$$\sup_n \max_{i,j \in \mathcal{N}_n} \left| \sum_{g_{i,-j}} D_x^q \sigma_i\big((1, g_{i,-j}) \,|\, X\big) \right| < \infty \quad \textit{and}$$
$$\sup_n \max_{i,j,k \in \mathcal{N}_n} \left| \sum_{g_{i,-jk}} D_x^q \sigma_i\big((1, 1, g_{i,-jk}) \,|\, X\big) \right| < \infty. \tag{1}$$

*(Note that the summands above correspond to $D_x^q \mathbf{E}[G_{ij} \,|\, X, \sigma]$ and $D_x^q \mathbf{E}[G_{ij} G_{ik} \,|\, X, \sigma]$, respectively.)*

Proof. Define $\Sigma^y, \Gamma$ as in the proof of Theorem 1. We will write $\Sigma_n^y \equiv \Sigma^y$ and $\Gamma_n \equiv \Gamma$ to emphasize their dependence on the number of agents. For $\sigma_n \in \Sigma_n^y$ and $X \in \mathbf{X}^n$, view $(\sigma_{ni}(a \mid X); a \in A^n)$ as the $i$th "row" of $\sigma_n(X)$. For each $n$, let $\tilde{\Sigma}_n^y \subseteq \Sigma_n^y$ such that for each $\sigma \in \tilde{\Sigma}_n^y$, (1) holds. These subsets are nonempty since constant functions satisfy these requirements. In view of Theorem 1, it only remains to show that $\Gamma_n(\cdot, X)$ maps $\tilde{\Sigma}_n^y$ to itself for all $n$ and that $\tilde{\Sigma}_n^y$ is closed. Regarding the first claim, notice that $\mathbf{E}[G_{ij} \mid X, \sigma] = \sum_{g_{i,-j}} D_x^q \Gamma(\sigma, X)_{i,(1,g_{i,-j})}$, where $\Gamma(\cdot, X)_{i,a}$ is the component of $\Gamma$ associated with agent $i$ and action $a \in A^n$ with $a_i = 0$. Then for any $\sigma \in \tilde{\Sigma}_n^y$ and $|q| = 1$,

$$
\begin{aligned}
D_x^q \mathbf{E}[G_{ij} \mid X, \sigma] &= D_x^q \Phi\big(Z_{ij}'\theta_0\big) \\
&= \Bigg( \sum_{g_{j,-i}} D_x^q \sigma_j\left((1, g_{j,-i}) \mid X\right), \frac{1}{n} \sum_{k \neq i} D_x^q \sum_{g_{k,-j}} \sigma_k\left((1, g_{k,-j}) \mid X\right) \mu(X_{ik}), \\
&\qquad \frac{1}{n} \sum_{k} \sum_{g_{i,-jk}} D_x^q \sigma_i\left((1, 1, g_{i,-jk}) \mid X\right), D_x^q X_{ij}' \Bigg) \theta_0 \times \phi(Z_{ij}'\theta_0),
\end{aligned}
$$

which is uniformly bounded by Assumption 1, 4, and the construction of $\tilde{\Sigma}_n^y$. Repeated application of the chain rule shows that this is also true for $q$ such that $|q| \in [1, s]$. Next, let $\Phi^{[2]}$ be the joint CDF of $(\varepsilon_{ij}, \varepsilon_{ik})$. Then for $|q| = 1$,

$$
\begin{aligned}
D_x^q \mathbf{E}[G_{ij} G_{ik} \mid X = x] &= D_x^q \Phi\big(-Z_{ij}'\theta_0, -Z_{ik}'\theta_0\big) \\
&= \Bigg( \sum_{g_{j,-i}} D_x^q \sigma_j\left((1, g_{j,-i}) \mid X\right), \frac{1}{n} \sum_{l \neq i} D_x^q \sum_{g_{l,-j}} \sigma_l\left((1, g_{l,-j}) \mid X\right) \mu(X_{il}), \\
&\qquad \frac{1}{n} \sum_{l} \sum_{g_{i,-jl}} D_x^q \sigma_i\left((1, 1, g_{i,-jl}) \mid X\right), D_x^q X_{ij}' \Bigg) \theta_0 \\
&\qquad \times D_1 \Phi\big(-Z_{ij}'\theta_0, -Z_{ik}'\theta_0\big) \\
&+ \Bigg( \sum_{g_{k,-i}} D_x^q \sigma_k\left((1, g_{k,-i}) \mid X\right), \frac{1}{n} \sum_{l \neq i} D_x^q \sum_{g_{l,-k}} \sigma_l\left((1, g_{l,-k}) \mid X\right) \mu(X_{il}), \\
&\qquad \frac{1}{n} \sum_{l} \sum_{g_{i,-kl}} D_x^q \sigma_i\left((1, 1, g_{i,-kl}) \mid X\right), D_x^q X_{ij}' \Bigg) \theta_0 \\
&\qquad \times D_2 \Phi\big(-Z_{ij}'\theta_0, -Z_{ik}'\theta_0\big),
\end{aligned}
$$

which is also uniformly bounded for the same reason. Repeated application of the chain rule shows that this is also true for $q$ such that $|q| \in [1, s]$. Hence, $\{\Gamma_n\}_{n=2}^{\infty}$ map, respectively, $\{\tilde{\Sigma}_n^y\}_{n=2}^{\infty}$ to themselves.

Lastly, note that uniform boundedness must carry over to limit points of $\tilde{\Sigma}_n^y$ for any $n$, so $\tilde{\Sigma}_n^y$ is closed. This completes the proof. ∎

This justifies our next assumption, which replaces Assumption 2 if kernel estimators are used. For each $n$, let $\mathcal{G}(X^n, \theta_0)$ be the set of $s$-times differentiable, symmetric equilibria such that any $\sigma^n \in \mathcal{G}(X^n, \theta_0)$ satisfies (1).

**Assumption A.1** (Selection Mechanism II)**.** *There exist sequences of equilibrium selection mechanisms $\{\lambda_n(\cdot); n \in \mathbb{N}\}$ and public signals $\{\nu^n; n \in \mathbb{N}\}$ such that for $n$ sufficiently large, $\mathcal{G}(X^n, \theta_0)$ is nonempty, and for any $g^n \in \mathbf{G}^n$,*

$$\mathbf{P}(G^n = g^n \,|\, X^n) = \sum_{\sigma^n \in \mathcal{G}(X^n, \theta_0)} \mathbf{P}\left(\lambda_n(X^n, \nu^n, \theta_0) = \sigma^n \,|\, X^n\right) \prod_{i=1}^{n} \sigma_i^n(g_i^n \,|\, X^n).$$

Next we demonstrate uniform consistency of the kernel estimator. We introduce multi-index notation for multivariate Taylor expansions for the following proof. Let $f : \mathbb{R}^m \to \mathbb{R}$ be continuously differentiable up to order $q$. For $\alpha \in \mathbb{N}^m$ and $x \in \mathbb{R}^m$, define

$$|\alpha| = \sum_{i=1}^{m} \alpha_i, \qquad \alpha! = \prod_{i=1}^{m} \alpha_i!, \qquad x^\alpha = \prod_{i=1}^{m} x_i^{\alpha_i}$$

and the $k$th order derivative

$$f^{(\alpha)}(x) = D^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}, \qquad |\alpha| \leq k.$$

**Theorem A.2.** *Under the following conditions,*

$$\sup_{i,j \in \mathcal{N}_n} \left| \frac{\sum_{k,l} G_{kl} K\left(\frac{X_{ij} - X_{kl}}{h}\right)}{\sum_{k,l} K\left(\frac{X_{ij} - X_{kl}}{h}\right)} - \mathbf{E}[G_{ij} \,|\, X, \sigma] \right| = O_p\left(h^s + \sqrt{\frac{\log n}{nh^d}}\right). \qquad (2)$$

*(a) $\{X_{ij}; i, j \in \mathcal{N}_n\}$ is identically distributed with an $s$-times differentiable density $p(\cdot)$.*

*(b) Assumption A.1 holds.*

*(c) The kernel $K : \mathbb{R}^d \to [0, 1]$ satisfies the following conditions.*

- *$K$ is a higher-order kernel of degree $2s$: $\int K(u)\, u = 1$; $\int u^r K(u)\, du = 0$ for all $r < s$; $\int |u^{2s}| K(u)\, du < \infty$; and $\int K^2(u)\, du < \infty$.*[1]
- *$K$ is* regular *in the sense of Einmahl and Mason (2005).*

*(d) $h \to 0$ and $nh^d \to \infty$.*

[1]Note that the order of the kernel is twice the smoothness of $p(\tilde{x})$ and $\mathbf{E}[G_{ij} \,|\, X, \sigma]$, unlike the standard case.

PROOF. We need some notation first. For $\tilde{x} \in \mathbf{X}$, define

$$\hat{p}(\tilde{x}) = \frac{1}{n^2 h^d} \sum_{k,l} K\left(\frac{\tilde{x} - X_{kl}}{h}\right),$$

the kernel estimate of $p(\tilde{x})$, and $X_{-ij}$ the matrix of attributes $X$ excluding $X_{ij}$. By a simple variance calculation, standard arguments yield $\sup_{\tilde{x}} |\hat{p}(\tilde{x}) - p(\tilde{x})| \xrightarrow{p} 0$.

By the triangle inequality, the left-hand side of equation (2) is at most

$$\underbrace{\sup_{i,j\in\mathcal{N}_n} \left| \frac{\sum_{k,l} \left(G_{kl} - \mathbf{E}[G_{kl} \mid X, \sigma]\right) K\left(\frac{X_{ij}-X_{kl}}{h}\right)}{\sum_{k,l} K\left(\frac{X_{ij}-X_{kl}}{h}\right)} \right|}_{\text{variance}}$$
$$+ \underbrace{\sup_{i,j\in\mathcal{N}_n} \left| \frac{\sum_{k,l} \mathbf{E}[G_{kl} \mid X, \sigma] K\left(\frac{X_{ij}-X_{kl}}{h}\right)}{\sum_{k,l} K\left(\frac{X_{ij}-X_{kl}}{h}\right)} - \mathbf{E}[G_{ij} \mid X, \sigma] \right|}_{\text{bias}}.$$

**Bias.** We show that the bias term is $O_p\left(h^s + \sqrt{\frac{\log n}{nh^d}}\right)$ uniformly in $i, j$. Fix $\sigma$ and $i, j \in \mathcal{N}_n$. By Proposition 1, $\mathbf{E}[G_{ij} \mid X, \sigma]$ is invariant to permutations of the components $X_{kl}$ of $X_{-ij}$. Then by Assumption A.1, for any $i, j \in \mathcal{N}_n$ and a given $\sigma$, we can write $\mathbf{E}[G_{ij} \mid X, \sigma] = \rho_n(X_i, X_j, X_{-ij})$, where $\rho_n$ is a function that is invariant to permutations of the components $X_{kl}$ of $X_{-ij}$. By the mean-value theorem,

$$\mathbf{E}[G_{kl} \mid X, \sigma] = \rho_n(\overbrace{X_{kl}, X_{ij}}, X_{-ij,kl}) = \underbrace{\rho_n(X_{ij}, X_{kl}, X_{-ij,kl})}_{\mathbf{E}[G_{ij} \mid X, \sigma]}$$
$$+ \sum_{0<|q|<s} D_2^q\, \mathbf{E}[G_{ij} \mid X, \sigma] \frac{1}{q!}(X_{kl} - X_{ij})^q + \sum_{|q|=s} D_2^q\, P^*_{ijkl} \frac{1}{q!}(X_{kl} - X_{ij})^q,$$

The term $P^*_{ijkl}$ lies between $\mathbf{E}[G_{kl} \mid X, \sigma]$ and $\mathbf{E}[G_{ij} \mid X, \sigma]$. The notation $D_2^q$ is meant to underscore the fact that the derivative is taken with respect to the two components highlighted by the overbracket. Then substituting in the Taylor expansions and using uniform consistency of $\hat{p}(X_{ij})$,

$$\left|\frac{\sum_{k,l}\mathbf{E}[G_{kl}\,|\,X,\sigma]K\left(\frac{X_{ij}-X_{kl}}{h}\right)}{\sum_{k,l}K\left(\frac{X_{ij}-X_{kl}}{h}\right)}-\mathbf{E}[G_{ij}\,|\,X,\sigma]\right|$$

$$\leq\left|\frac{1}{\hat{p}(X_{ij})}\underbrace{\frac{1}{n^2}\sum_{k,l}\frac{1}{h^d}K\left(\frac{X_{ij}-X_{kl}}{h}\right)}_{\hat{p}(X_{ij})}\mathbf{E}[G_{ij}\,|\,X,\sigma]-\mathbf{E}[G_{ij}\,|\,X,\sigma]\right|$$

$$+\left|\frac{1}{\hat{p}(X_{ij})}\frac{1}{n^2}\sum_{k,l}\frac{1}{h^d}K\left(\frac{X_{ij}-X_{kl}}{h}\right)\sum_{0<|q|<s}D_2^q\,\mathbf{E}[G_{ij}\,|\,X,\sigma]\frac{1}{q!}\tilde{(}X_{kl}-X_{ij})^q\right|$$

$$+\left|\frac{1}{\hat{p}(X_{ij})}\frac{1}{n^2}\sum_{k,l}\frac{1}{h^d}K\left(\frac{X_{ij}-X_{kl}}{h}\right)\sum_{|q|=s}D_2^q\,P^*_{ijkl}\frac{1}{q!}\tilde{(}X_{kl}-X_{ij})^q\right|$$

$$\leq\sum_{0<|q|\leq s}B_q\frac{1}{q!}\left|\underbrace{\frac{1}{n^2}\sum_{k,l}\frac{1}{h^d}K\left(\frac{X_{ij}-X_{kl}}{h}\right)\tilde{(}X_{kl}-X_{ij})^q\frac{1}{p(X_{ij})}}_{\mathbf{A}}\right|(1+o_p(1)),\quad (3)$$

where $B_q<\infty$ and $\sup_n\max_{i,j\in\mathcal{N}_n}D_2^q<B_q$. This exists by condition (b). By standard arguments, the supremum of $\mathbf{A}$ over all $X_{ij}$ uniformly converges to its expectation at the rate $O(\sqrt{(nh^d)^{-1}\log n})$. Moreover, this expectation is $O(h^s)$ by the usual change of variables $u=\frac{X_{kl}-X_{ij}}{h}$. This establishes the bound on the bias.

**Variance.** We next show that the variance term is $O_p\left(\sqrt{\frac{\log n}{nh^d}}\right)$. By equation (4),

$$G_{ij}=\mathbf{1}\left\{Z_{ij}'\theta_0+\varepsilon_{ij}>0\right\},$$

which is a function of random elements $\varepsilon_{ij}$ and $X$. Since $\varepsilon\perp\!\!\!\perp X$ by Assumption 1, conditioning on $X$ is the same as fixing attributes and the equilibrium $\sigma$ (by Assumption A.1) and dropping the conditioning, so we can apply the usual empirical process results for independent data. Hence, in what follows, we treat $X,\sigma$ as fixed.

Notice that the variance term is at most

$$\sup_{\tilde{x}}\left|\frac{\sum_{k,l}\left(G_{kl}-\mathbf{E}[G_{kl}\,|\,X,\sigma]\right)K\left(\frac{\tilde{x}-X_{kl}}{h}\right)}{\sum_{k,l}K\left(\frac{\tilde{x}-X_{kl}}{h}\right)}\right|$$

$$\leq\underbrace{\sup_{\tilde{x}}\left|\frac{1}{n^2h^d}\sum_{k,l}\left(G_{kl}-\mathbf{E}[G_{kl}\,|\,X,\sigma]\right)K\left(\frac{\tilde{x}-X_{kl}}{h}\right)\right|}_{\mathbf{B}}\underbrace{\sup_{\tilde{x}}\left|\frac{1}{\hat{p}(\tilde{x})}\right|}_{\mathbf{C}}.$$

By a simple variance calculation, standard arguments and Assumption 1 yield $\mathbf{C}=O_p(1)$ by (a), so it remains to compute the rate of convergence of $\mathbf{B}$. We first need several definitions. Define

$$r_{kl}(\tilde{x},h) \equiv r(Z_{ij}'\theta_0, X_{kl}, \varepsilon_{kl}; \tilde{x}, h) = K\left(\frac{\tilde{x}-X_{kl}}{h}\right)\mathbf{1}\left\{\varepsilon_{kl} \geq Z_{ij}'\theta_0)\right\},$$

$\mathcal{R} = \{r(\cdot,\cdot,\cdot;\tilde{x},h)\}$ and $\eta = M\sqrt{\frac{\log n}{nh^d}}$ for any $M > 0$. Let $\xi_1,\ldots,\xi_n$ be i.i.d. Rademacher random variables, independent of $X, \varepsilon$.[2] For a given $\eta$, let $N_1\left(\frac{1}{8}\eta, \mathbf{P}_n, \mathcal{R}\right)$ be the smallest $m$ for which there exist functions $r_{[1]},\ldots,r_{[m]}$ such that $\min_j \mathbf{P}_n|r - r_{[j]}| < \frac{1}{8}\eta$ for all $r \in \mathcal{R}$. Then by a standard symmetrization argument (see e.g. Pollard, 1984, (11) and Theorem 24),

$$\begin{aligned}
&\mathbf{P}\left(\sup_{\tilde{x}}\left|\frac{1}{nh^d}\sum_k\left(\frac{1}{n}\sum_l r_{kl}(\tilde{x},h) - \mathbf{E}\left[\frac{1}{n}\sum_l r_{kl}(\tilde{x},h)\,\middle|\,X\right]\right) > \eta\,\right|X,\sigma\right) \qquad (4)\\
&\leq 4\,\mathbf{P}\left(\sup_{\tilde{x}}\left|\frac{1}{nh^d}\sum_k \xi_k\left(\frac{1}{n}\sum_l r_{kl}(\tilde{x},h)\right) > \frac{1}{4}\eta\,\right|X,\sigma\right)\\
&\leq 4\,N_1\left(\frac{1}{8}\eta,\mathbf{P}_n,\mathcal{R}\right)\times\\
&\quad \max_{j\in\{1,\ldots,m\}}\mathbf{P}\left(\left|\frac{1}{nh^d}\sum_k\xi_k\left(\frac{1}{n}\sum_l r_{[j]}(Z_{kl}'\theta_0,X_{kl},\varepsilon_{kl})\right)\right| > \frac{1}{8}\eta\,\middle|\,X,\sigma\right).
\end{aligned}$$

Since $K(\cdot)$ is bounded due to Assumption 1,

$$\left|\frac{1}{nh^d}\xi_k\left(\frac{1}{n}\sum_l r_{[j]}(Z_{kl}'\theta_0,X_{kl},\varepsilon_{kl};\tilde{x},h)\right)\right| < C\frac{1}{nh^d}$$

for some $C > 0$. In view of applying Bernstein's inequality (e.g. Pollard, 1984, Appendix B), we next show that

$$\operatorname{Var}\left(\frac{1}{nh^d}\sum_k\xi_k\left(\frac{1}{n}\sum_l r_{[j]}(Z_{kl}'\theta_0,X_{kl},\varepsilon_{kl};\tilde{x},h)\right)\,\middle|\,X,\sigma\right) < D\frac{1}{nh^d} \qquad (5)$$

for some $D > 0$ with probability approaching one. For any $\tilde{x} \in \mathbf{X}$, the left-hand side

[2] That is, $\mathbf{P}(\xi{=}1) = \mathbf{P}(\xi{=}-1) = 0.5$.

is at most

$$\frac{1}{n^4h^{4d}}\sum_{k,l}K^2\left(\frac{\tilde{x}-X_{kl}}{h}\right)\mathbf{E}[G_{kl}\,|\,X]\left(1-\mathbf{E}[G_{kl}\,|\,X]\right)$$
$$+\frac{1}{n^4h^{4d}}\sum_{k,l,m}K\left(\frac{\tilde{x}-X_{kl}}{h}\right)K\left(\frac{\tilde{x}-X_{km}}{h}\right)\mathrm{Cov}(G_{kl},G_{km}\,|\,X)$$
$$\leq\frac{1}{4n^2h^d}\underbrace{\frac{1}{n^2h^d}\sum_{k,l}K^2\left(\frac{\tilde{x}-X_{kl}}{h}\right)}_{\mathbf{D}}$$
$$+\frac{1}{nh^d}\underbrace{\frac{1}{n^3h^d}\sum_{k,l,m}K\left(\frac{\tilde{x}-X_{kl}}{h}\right)K\left(\frac{\tilde{x}-X_{km}}{h}\right)}_{\mathbf{E}}.$$

By (c),

$$\mathbf{D}=\frac{1}{n}\sum_{k}\frac{1}{h^d}K^2\left(\frac{\tilde{x}_1-X_k}{h}\right)\times\frac{1}{n}\sum_{l}\frac{1}{h^d}K^2\left(\frac{\tilde{x}_2-X_l}{h}\right)\quad\text{and}$$
$$\mathbf{E}\leq\sup_u K^2(u)\times\underbrace{\frac{1}{n}\sum_{l}\frac{1}{h^d}K\left(\frac{\tilde{x}_2-X_l}{h}\right)\times\frac{1}{n}\sum_{m}\frac{1}{h^d}K\left(\frac{\tilde{x}_2-X_m}{h}\right)}_{\hat{p}(\tilde{x})}.$$

As argued previously, these averages converge to their expectations uniformly in $\tilde{x}$ at rate $\sqrt{(nh^d)^{-1}\log n}$. After a change of variables, the expectations associated with $\mathbf{B}$ take the form $\int K^2(u)p(u+\tilde{x}h)du<\sup_{\tilde{x}}p(\tilde{x})\int K^2(u)du$. This is finite by Assumptions 1 and (c). Meanwhile, $\sup_u K^2(u)<\infty$, since $K$ is bounded by. This establishes (5).

Thus, by Bernstein's inequality, with probability approaching one,

$$\max_j \mathbf{P}\left(\left|\frac{1}{nh^d}\sum_k\xi_k\left(\frac{1}{n}\sum_l r_{[j]}(Z'_{kl}\theta_0,X_{kl},\varepsilon_{kl})\right)\right|>\frac{1}{8}\eta\,\middle|\,X,\sigma\right)$$
$$\leq 2\exp\left\{-\frac{1}{2}\frac{\eta^2}{\frac{1}{nh^d}D+\frac{1}{3nh^d}C\eta}\right\}$$
$$\leq 2\exp\left\{-\frac{1}{2}\frac{\frac{\log n}{nh^d}M^2}{\frac{D}{nh^d}+\frac{C}{3nh^d}\sqrt{\frac{\log n}{nh^d}}M}\right\}.$$

which is $O\left(\exp\{-M^2\log n\}\right)$. Further note that functions in $\mathcal{R}$ are products of indicators which do not depend on $(\tilde{x},h)$ and regular kernels by (c). By definition, the class of such kernels is VC-subgraph. Hence, $\mathcal{R}$ is a VC-subgraph class by Lemma

2.6.18(vi) of van der Vaart and Wellner (1996). By Theorem 2.6.7 of van der Vaart and Wellner (1996), there exists a constant $K$ such that

$$\log N_1\left(\frac{1}{8}\eta, \mathbf{P}_n, \mathcal{R}\right) = \log\left(KV(\mathcal{R})(16e)^{V(\mathcal{R})} - (V(\mathcal{R})-1)(\log\eta - \log 8)\right),$$

where $V(\mathcal{R})$ is the VC-index of $\mathcal{R}$. This is $O(\log n)$ by definition of $\eta$. Thus,

$$(4) = O\left(n^{-M^2}\log n\right) \overset{n\to\infty}{\longrightarrow} 0,$$

for any $M > 0$ a.s. This completes the proof. ∎

With some minor changes, the same argument can be applied to compute the uniform rate of convergence for analogous kernel estimators for supported trust and weighted in-degree. For example, for weighted-in-degree, we again use the trick in Proposition 2 to rewrite the numerator of the kernel estimator as a sum over out-degree:

$$\frac{1}{n^2h^d}\sum_{i,j\neq i}\mu(X_{ij})\left(\frac{1}{n}\sum_k G_{jk}K\left(\frac{\tilde{x}-X_{ik}}{h}\right)\right).$$

Then the main change to the consistency proof is to instead define

$$r_{kl}(\tilde{x},h) \equiv r(Z'_{jk}\theta_0, X_{ik}, X_{ij}, \varepsilon_{jk}; \tilde{x}, h) = K\left(\frac{\tilde{x}-X_{ik}}{h}\right)\mathbf{1}\left\{\varepsilon_{jk} \geq Z'_{jk}\theta_0)\right\}\mu(X_{ij}).$$

**Theorem A.3** (Asymptotic Normality). *Assume the following.*

*(a) The assumptions of Theorem 3 hold.*[3]

*(b) The assumptions of Theorem A.2 hold for $s > d$.*

*Then* $\sqrt{n}\Lambda_n^{-1/2}(\hat{\theta}-\theta_0) \overset{d}{\longrightarrow} N(0, I_p)$*, where* $\Lambda_n = \mathcal{V}_n^{-1}\Psi_n\mathcal{V}_n^{-1}$*, defined in Theorem 3.*[4]

Proof. Define $M_{ij}$ as in the proof of Theorem 3. It is enough to show that equation (23) of Theorem 3 is $o_p(1)$; the rest of the proof remains the same. Since $\hat{p}(\tilde{x})$ is uniformly consistent for $p(\tilde{x})$, which is bounded away from zero on its support by

[3]As usual, condition (b) of Theorem 3 implies $s > d/2$.

[4]Recall that $p$ is the dimension of $Z_{ij}$.

assumption,

$$
\begin{aligned}
(23) =& \frac{1}{n\sqrt{n}} \sum_{k,l} \left( \frac{1}{n^2 h^d} \sum_{i,j} \frac{M_{ij}}{\hat{p}(X_{ij})} K\left(\frac{X_{ij}-X_{kl}}{h}\right) - M_{kl} \right) \mathcal{E}_{kl} \\
=& \underbrace{\frac{1}{n\sqrt{n}} \sum_{k,l} \left( \frac{1}{n^2 h^d} \sum_{i,j} \frac{M_{ij}}{p(X_{ij})} K\left(\frac{X_{ij}-X_{kl}}{h}\right) - M_{kl} \right) \mathcal{E}_{kl}}_{\mathbf{A}} \big(1 + o_p(1)\big) \\
&+ \underbrace{\left( \frac{1}{n\sqrt{n}} \sum_{k,l} M_{kl}\, \mathcal{E}_{kl} \right)}_{\mathbf{B}} o_p(1).
\end{aligned}
$$

It remains to show that $\mathbf{A}$ is $o_p(1)$ and $\mathbf{B}$ is $O_p(1)$. We can rewrite $\mathbf{B}$ as

$$
\frac{1}{\sqrt{n}} \sum_{l} \left( \frac{1}{n} \sum_{k} M_{kl}[1] G_{lk} \right) + \frac{1}{\sqrt{n}} \sum_{m} \left( \frac{1}{n^2} \sum_{k,l} M_{kl}[-1] \begin{pmatrix} G_{ml}\mu(X_{km}) \\ G_{mk} G_{ml} \end{pmatrix} \right),
$$

where $M_{kl}[1]$ is the first column of $M_{kl}$, and $M_{kl}[-1]$ is all but the first column of $M_{kl}$. Since the summands are independent conditional on $X$ and uniformly bounded by Assumptions 1 and 4, Lyapunov's CLT applies, and $\mathbf{B}$ is $O_p(1)$ as desired.

By a similar argument, $\mathbf{A}$ is $o_p(1)$ if $\frac{1}{n^2 h^d} \sum_{i,j} \frac{M_{ij}}{p(X_{ij})} K\left(\frac{X_{ij}-X_{kl}}{h}\right) - M_{kl}$ is $o_p(1)$ uniformly in $k, l$. To show the latter, we take a Taylor expansion of $M_{kl}$. By the same argument as equation (3) in the proof of Theorem A.2,

$$
\begin{aligned}
&\left| \frac{1}{n^2 h^d} \sum_{i,j} \left[ \frac{M_{ij}}{p(X_{ij})} K\left(\frac{X_{ij}-X_{kl}}{h}\right) - M_{kl} \right] \right| \\
&\qquad = \left| \frac{1}{n^2 h^d} \sum_{i,j} \left[ \frac{M_{kl}}{p(X_{ij})} K\left(\frac{X_{ij}-X_{kl}}{h}\right) - M_{kl} \right] \right| \\
&\qquad + \left| \frac{1}{n^2 h^d} \sum_{i,j} K\left(\frac{X_{ij}-X_{kl}}{h}\right) \sum_{0<|q|<s} \frac{D_2^q\, M_{kl}}{p(X_{ij})} \frac{1}{q!} \tilde{(}X_{kl} - X_{ij})^q \right| \\
&\qquad + \left| \frac{1}{n^2 h^d} \sum_{i,j} K\left(\frac{X_{ij}-X_{kl}}{h}\right) \sum_{|q|=s} \frac{D_2^q\, \tilde{M}^*_{ijkl}}{p(X_{ij})} \frac{1}{q!} \tilde{(}X_{kl} - X_{ij})^q \right| \\
&\qquad \le \left| \frac{1}{n^2 h^d} \sum_{i,j} \frac{1}{p(X_{ij})} K\left(\frac{X_{ij}-X_{kl}}{h}\right) - 1 \right| B_0 \\
&\qquad + \sum_{0<|q|\le s} B_q \frac{1}{q!} \left| \frac{1}{n^2} \sum_{i,j} \frac{1}{h^d} K\left(\frac{X_{ij}-X_{kl}}{h}\right) \frac{1}{p(X_{ij})} \tilde{(}X_{kl} - X_{ij})^q \right|,
\end{aligned}
$$

where $B_q < \infty$, $\sup_n \max_{i,j\in\mathcal{N}_n} |D_2^q M_{ij}| < B_q$ for all $n$, and $D_2^q$ is defined in the proof of Theorem A.2.[5] Since $K(\cdot)$ is regular by condition (c) of Theorem A.2, the uniform entropy condition of Theorem 2.4.3 of van der Vaart and Wellner (1996) holds (see the argument at the end of the proof of Theorem A.2). Hence, Theorem 2.4.3 implies that the absolute-value terms in the last line converge to their expectations uniformly in $X_{kl}$ and $h$. In particular,

$$\sup_{h,X_{kl}} \left| \frac{1}{n^2} \sum_{i,j} \frac{1}{h^d} K\left(\frac{X_{ij}-X_{kl}}{h}\right) \frac{1}{p(X_{ij})} \tilde{(}X_{kl}-X_{ij})^q \right. \\ \left. - \int_{X_{ij}} \frac{1}{h^d} K\left(\frac{X_{ij}-X_{kl}}{h}\right) \tilde{(}X_{kl}-X_{ij})^q \,\mathrm{d}X_{ij} \right| \xrightarrow{p} 0.$$

As argued in the proof of Theorem A.2, the integral on the right-hand side is $O(h^s)$ uniformly in $X_{kl}$ by a change of variables $u = \frac{X_{ij}-X_{kl}}{h}$. Similarly,

$$\frac{1}{n^2h^d} \sum_{i,j} \frac{1}{p(X_{ij})} K\left(\frac{X_{ij}-X_{kl}}{h}\right) - 1 \\ = \underbrace{\frac{1}{n^2h^d} \sum_{i,j} \frac{1}{p(X_{ij})} K\left(\frac{X_{ij}-X_{kl}}{h}\right) - \int_{X_{ij}} \frac{1}{h^d} K\left(\frac{X_{ij}-X_{kl}}{h}\right) \mathrm{d}X_{ij}}_{o_p(1)} \\ + \underbrace{\int_{X_{ij}} \frac{1}{h^d} K\left(\frac{X_{ij}-X_{kl}}{h}\right) \mathrm{d}X_{ij} - 1}_{0}.$$

This completes the proof. ■

# B Additional Application Results

The tables below summarize coefficient estimates and standard errors that are not presented in the main text.

[5] This derivative exists by condition Theorem A.2, condition (b). It is uniformly bounded by Assumptions 1 and 4.

Table 1: Controls for $i$ (ego).

| | $\lambda = 0$ | $\lambda = 0.1$ | $\lambda = 0.2$ | dyadic |
|---|---|---|---|---|
| head of household | $0.08^{*}$ | $0.08^{*}$ | $0.08^{*}$ | $0.09^{***}$ |
| | (0.05) | (0.05) | (0.05) | (0.03) |
| OBC caste | -0.04 | -0.04 | -0.04 | $-0.08^{**}$ |
| | (0.08) | (0.08) | (0.09) | (0.03) |
| scheduled caste | -0.04 | -0.04 | -0.04 | -0.05 |
| | (0.07) | (0.08) | (0.08) | (0.03) |
| female | $0.11^{*}$ | $0.12^{*}$ | $0.11^{*}$ | $0.11^{***}$ |
| | (0.06) | (0.06) | (0.06) | (0.03) |
| hindu | -0.06 | -0.06 | -0.05 | $-0.06^{**}$ |
| | (0.05) | (0.05) | (0.05) | (0.03) |

Note: Standard errors are in parentheses. (*) denotes significance at the 10% level, (**) the 5% level, and (***) the 1% level.

Table 2: Controls for $j$ (alter).

| | $\lambda = 0$ | $\lambda = 0.1$ | $\lambda = 0.2$ | dyadic |
|---|---|---|---|---|
| head of household | 0.02 | 0.03 | 0.03 | 0.04 |
| | (0.07) | (0.07) | (0.08) | (0.03) |
| OBC caste | -0.12 | -0.12 | -0.12 | $-0.16^{***}$ |
| | (0.09) | (0.10) | (0.11) | (0.03) |
| scheduled caste | -0.04 | -0.05 | -0.04 | $-0.09^{***}$ |
| | (0.10) | (0.11) | (0.12) | (0.03) |
| female | -0.02 | -0.01 | 0.00 | 0.02 |
| | (0.09) | (0.09) | (0.10) | (0.03) |
| hindu | -0.07 | -0.06 | -0.05 | -0.03 |
| | (0.07) | (0.08) | (0.09) | (0.03) |

Note: Standard errors are in parentheses. (*) denotes significance at the 10% level, (**) the 5% level, and (***) the 1% level.

## References

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