Supplementary Appendix for "A Weak Law for Moments of Pairwise Stable Networks"

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February 11, 2019

SA.1 General Model and Coupling

This section defines the general model under which we derive all results in the appendix. Unlike §4, we allow attributes to be correlated with positions. The notation used here and in the remainder of the appendix differs in important aspects from the main text. This is because we need a sufficiently general setup that can accommodate both the finite-n model (Assumption 3) and the limit model in §4 where the number of nodes is infinite, in order to explicitly characterize limiting network moments.

Positions. We construct a coupling between the sets of node positions in the finite and limit models, namely $\{\mathcal{X}_n : n \in \mathbb{N}\}$ and $\mathcal{P}_{\kappa f(X)}$, following Penrose and Yukich (2003). Let X be a random vector with density f, where f is continuous and bounded above on $\mathbb{R}^d$. Let $\mathcal{P}_1$ be a Poisson point process of rate one on $\mathbb{R}^d \times [0, \infty)$ independent of X.

- Finite Model: Let $\mathcal{P}^*_{nf}$ be the restriction of $\mathcal{P}_1$ to the set

$$\{(x, t) \in \mathbb{R}^d \times [0, \infty) : t \leqslant nf(x)\}$$

  and $\mathcal{P}_{nf}$ the image of $\mathcal{P}^*_{nf}$ under the projection $(x, t) \mapsto x$ for $x \in \mathbb{R}^d$ and $t \in [0, \infty)$. Then $\mathcal{P}_{nf}$ is an inhomogeneous Poisson point process with intensity function $nf(\cdot)$ by the mapping theorem for Poisson processes (e.g. Kingman, 1993).

  Let $N_n = |\mathcal{P}_{nf}|$, which is a Poisson random variable with intensity n (Penrose, 2003, Proposition 1.5). We remove from $\mathcal{P}_{nf}$ a set of $\max\{N_n - (n-1), 0\}$ elements chosen at random and add $\max\{n - 1 - N_n, 0\}$ random vectors drawn i.i.d. from f. Call the

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resulting set $\mathcal{X}'_{n-1}$, and define $\mathcal{X}_n = \mathcal{X}'_{n-1} \cup \{X\}$. By construction, this is a set of n random vectors drawn i.i.d. from f.

- Limit Model: For any $x \in \mathbb{R}^d$, let $\mathcal{P}^*_{nf(x)}$ be the restriction of $\mathcal{P}_1$ to

$$\{(y,t) \in \mathbb{R}^d \times [0,\infty) : t \leqslant nf(x)\}$$

and $\mathcal{P}^n_{\kappa f(x)}$ the image of $\mathcal{P}^*_{nf(x)}$ under $(y,t) \mapsto x + r_n^{-1}(y-x)$. Then for all n, $\mathcal{P}^n_{\kappa f(x)}$ is a Poisson point process on $\mathbb{R}^d$ with intensity $\kappa f(x)$ by the mapping theorem. We define $\mathcal{P}_{\kappa f(x)} = \mathcal{P}^\kappa_{\kappa f(x)}$.

Attributes. In the main text, we write node statistics as functionals of W, which is a set of attributes. We can equivalently define W as a random mapping from pairs of node positions to an attribute vector. The latter characterization is more convenient to allow for arbitrary sets of node positions with possibly infinite elements. We next define this *attribute process* $\mathbf{W}$. Let $\tilde{Z}$ be a stochastic process on $\mathbb{R}^d$ such that $\{\tilde{Z}(x); x \in \mathbb{R}^d\}$ is i.i.d. and ζ a stochastic process on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\{\zeta(x,x'); x, x' \in \mathbb{R}^d\}$ is i.i.d. Assume $\tilde{Z} \perp\!\!\!\perp \zeta \perp\!\!\!\perp \mathcal{P}_1, X, Y$. Let $H_z : \mathbb{R}^{d+d_{\tilde{z}}} \to \mathbb{R}^{d_z}$. Define Z as a stochastic process on $\mathbb{R}^d$ such that $Z(x) = H_z(x, \tilde{Z}(x))$, which represents the node-level attribute associated with the node positioned at x. Note that this construction is without loss of generality and allows for arbitrary dependence between a node's attribute vector and her position. Likewise, $\zeta(x,x')$ is the pair-level attribute associated with two nodes respectively positioned at x and x'.

For any $\mathcal{X} \subseteq \mathbb{R}^d$, the attribute process $\mathbf{W}(\mathcal{X}) \equiv \mathbf{W}(\cdot,\cdot;\mathcal{X})$ is the mapping from $\mathcal{X} \times \mathcal{X}$ to $\mathbb{R}^{d_z} \times \mathbb{R}^{d_z} \times \mathbb{R}^{d_\zeta}$ given by

$$\mathbf{W}(y,y';\mathcal{X}) = \big(H_z(y,\tilde{Z}(y)), H_z(y',\tilde{Z}(y')), \zeta(y,y')\big).$$

Let $\Phi(\cdot \,|\, x)$ be the distribution of $H_z(x, \tilde{Z}(x))$.

Dilation. We often take a "composition" of $\mathbf{W}$ and certain operators. This makes for compact notation that simplifies the expression of the limiting moment. Let $r \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$. Define the *dilation operator* $\tau_{x,r} : y \mapsto x + r^{-1}(y-x)$ and its inverse, the *contraction operator*, $\mathbf{p}_{x,r} : y \mapsto (y-x)r + x$. Abusing notation, we let $\tau_{x,r}\mathcal{X} = \{x + r^{-1}(y-x) : y \in \mathcal{X}\}$ and $\mathbf{p}_{x,r}\mathcal{X} = \{\mathbf{p}_{x,r}(y) : y \in \mathcal{X}\}$. These operators will be used extensively in the proofs. To reduce notation, we will sometimes write $\tau_{x,r}\mathcal{X} \cap S$ to mean $\tau_{x,r}(\mathcal{X}) \cap S$. That is, the order of operations is to prioritize $\tau_{x,r}$ before set operations.

Define the stochastic processes $\mathbf{W}\mathbf{p}_{x,r}$ and $\mathbf{W}^*\mathbf{p}_{x,r}$ on $\mathbb{R}^d \times \mathbb{R}^d$ where for any $y, y' \in \mathbb{R}^d$,

$$\mathbf{W}\mathbf{p}_{x,r}(y,y') = \big(H_z(\mathbf{p}_{x,r}y, \tilde{Z}(\mathbf{p}_{x,r}y)), H_z(\mathbf{p}_{x,r}y', \tilde{Z}(\mathbf{p}_{x,r}y')), \zeta(\mathbf{p}_{x,r}y, \mathbf{p}_{x,r}y')\big),$$
$$\mathbf{W}^*\mathbf{p}_{x,r}(y,y') = \big(H_z(\mathbf{p}_{x,r}y, \tilde{Z}(y)), H_z(\mathbf{p}_{x,r}y', \tilde{Z}(y')), \zeta(y,y')\big).$$

Observe that the corresponding finite dimensional distributions of the two processes are identical, since $\{\tilde{Z}(x); x \in \mathbb{R}^d\}$ is i.i.d.. For instance, $H_z(\mathbf{p}_{x,r}y, \tilde{Z}(\mathbf{p}_{x,r}y))$ and $H_z(\mathbf{p}_{x,r}y, \tilde{Z}(y))$ have the same distribution $\Phi(\cdot \mid \mathbf{p}_{x,r}y)$.

LLN. The appendix primarily considers network-formation models of the form

$$(V, \lambda, \tau_{x,r}\mathcal{X}, \mathbf{W}\mathbf{p}_{x,r}, r')$$

for $\mathcal{X} \subseteq \mathbb{R}^d$, $x \in \mathbb{R}^d$, and $r, r' \geqslant 0$. Note that by Assumption 2, the network A is a deterministic functional of the set of node positions, the attribute process, and the sparsity parameter. As in §4, we abbreviate

$$\psi(X, \tau_{x,r}\mathcal{X}, \mathbf{W}\mathbf{p}_{x,r}, A(r')) \equiv \psi(X, \tau_{x,r}\mathcal{X}, \mathbf{W}\mathbf{p}_{x,r}(\tau_{x,r}\mathcal{X}), A(\tau_{x,r}\mathcal{X}, \mathbf{W}\mathbf{p}_{x,r}(\tau_{x,r}\mathcal{X}), r')).$$

Note that in the notation of the main text, the node statistic for a node positioned at $X \in \mathcal{X}_n$ in the finite model can be rewritten using our new notation as

$$\psi\big(X, \mathcal{X}_n, W, A(r_n)\big) = \psi\big(X, \mathcal{X}_n, \mathbf{W}, A(r_n)\big) = \psi\big(X, \tau_{X,r_n}\mathcal{X}_n, \mathbf{W}\mathbf{p}_{X,r_n}, A(1)\big). \tag{SA.1.1}$$

Theorem 1 assumes positions are independent of attributes. In the notation of this section, this means H_z does not vary in its first argument. The following assumption relaxes this requirement.

Assumption 8 (Continuity)**.** *(a) The probability mass/density function associated with $\Phi(\cdot \mid x)$ is continuous in x. (b) $\mathbf{E}[\psi(X, \mathcal{P}_{\kappa f(X)} \cup \{X\}, \mathbf{W}^*\mathbf{p}_{X,r}, A(1))]$ is continuous at $r = 0$.*

Part (b) implicitly imposes a high-level continuity condition on the primitives, including the selection mechanism. In §SA.5.2, we show for the leading case of subgraph counts that this condition holds under part (a) of the previous assumption and a continuity condition on λ.

Theorem 2 (Weak Law)**.** *Suppose there exists $\epsilon > 0$ such that (14) holds. Then under*

Assumptions 1–8, as $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n} \psi\big(X_i, \mathcal{X}_n, W, A(r_n)\big) \xrightarrow{L_2} \int \mathbf{E}\left[\psi\left(x, \mathcal{P}_{\kappa f(x)} \cup \{x\}, \mathbf{W}^*\mathbf{p}_{x,0}, A(1)\right)\right] f(x)\, dx.$$

Proof. See §SA.4. ∎

SA.2 Weak Law for Stabilizing Functionals

We maintain the coupling construction and notation introduced in §SA.1. For any $x \in \mathbb{R}^d$, $r \in \mathbb{R}_+$, and $\mathcal{X} \subseteq \mathbb{R}^d$, let $\xi(x, \mathcal{X}, \mathbf{W}\mathbf{p}_{x,r})$ be a real-valued functional. We omit the argument of $\mathbf{W}\mathbf{p}_{x,r}$ with the understanding that it equals the second argument of ξ. Additionally, if $x \notin \mathcal{X}$, we let $\xi(x, \mathcal{X}, \mathbf{W}\mathbf{p}_{x,r}) \equiv \xi(x, \mathcal{X} \cup \{x\}, \mathbf{W}\mathbf{p}_{x,r})$.

We modify Theorem 2.1 of Penrose and Yukich (2003) for our setting in three ways. First, we use a different notion of stabilization. Second, we allow ξ to depend on $\mathbf{W}$ that is not identically distributed conditional on positions (in contrast to their definition of marked point processes). Third, we relax their translation-invariance assumption on ξ.

Recall that $B(x, r)$ denotes the ball of radius r in $\mathbb{R}^d$ with center x.

Definition 4. The functional ξ is *stabilizing* if there exist non-negative random variables $\{\mathbf{R}_n\}_{n\in\mathbb{N}}$ such that (a) $\{\mathbf{R}_n\}_{n\in\mathbb{N}}$ is bounded in probability, and (b) for all n sufficiently large and $R \geqslant \mathbf{R}_n$, with probability one,

$$\xi\big(X, \tau_{X,r_n}\mathcal{X}_n, \mathbf{W}\mathbf{p}_{X,r_n}\big) = \xi\big(X, \tau_{X,r_n}\mathcal{X}_n \cap B(X,R), \mathbf{W}\mathbf{p}_{X,r_n}\big), \tag{SA.2.1}$$

and

$$\xi\big(X, \mathcal{P}_{\kappa f(X)}, \mathbf{W}\mathbf{p}_{X,r_n}\big) = \xi\big(X, \mathcal{P}_{\kappa f(X)} \cap B(X,R), \mathbf{W}\mathbf{p}_{X,r_n}\big). \tag{SA.2.2}$$

Part (b) says that ξ is invariant to the removal of nodes outside of the neighborhood $B(X, \mathbf{R}_n)$. The neighborhood size is asymptotically bounded by part (a). Equation (SA.2.2) is a variant of the notion of strong stabilization in Penrose and Yukich (2003). Their definition additionally demands invariance to the *addition* of points outside of the neighborhood $B(x, \mathbf{R}_n)$, which is not suitable for our application. In the notation of §5, adding new nodes to $\mathcal{N}_n \backslash J_i$ can well change the value of the node statistic, since it may be attractive to link with the new nodes. Another difference is that Penrose and Yukich (2003) only require stabilization under the Poisson point process, whereas we also require equation (SA.2.1), which is a variant of "binomial stabilization" (e.g. Penrose, 2007) ($\mathcal{X}_n$ is referred to the binomial process in that literature).

Theorem 3. *Suppose ξ is stabilizing,*

$$\sup_{n\in\mathbb{N}} \mathbf{E}\left[\xi(X, \tau_{X,r_n}\mathcal{X}_n, \mathbf{Wp}_{X,r_n})^{2+\epsilon}\right] < \infty \tag{SA.2.3}$$

for some $\epsilon > 0$, and $\mathbf{E}[\xi(X, \mathcal{P}_{\kappa f(X)}, \mathbf{W}^*\mathbf{p}_{X,r})]$ is continuous at $r = 0$. Then as $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n} \xi(X_i, \tau_{X_i,r_n}\mathcal{X}_n, \mathbf{Wp}_{X_i,r_n}) \xrightarrow{L_2} \int \mathbf{E}\left[\xi\big(x, \mathcal{P}_{\kappa f(x)}, \mathbf{Wp}_{x,0}\big)\right] f(x)\, dx. \tag{SA.2.4}$$

The proof of this theorem requires the following two lemmas.

Lemma 3 (Coupling)**.** *For any $R > 0$,*

$$\lim_{n\to\infty} \mathbf{P}\left(\tau_{X,r_n}\mathcal{X}'_{n-1} \cap B(X,R) = \mathcal{P}^n_{\kappa f(X)} \cap B(X,R)\right) = 1. \tag{SA.2.5}$$

Proof. This is a restatement of Lemma 3.1 of Penrose and Yukich (2003). ■

Lemma 4 (Convergence of Means)**.** *Under the assumptions of Theorem 3,*

$$\mathbf{E}\left[\xi\big(X, \tau_{X,r_n}\mathcal{X}'_{n-1}, \mathbf{Wp}_{X,r_n}\big)\right] \to \mathbf{E}\left[\xi\big(X, \mathcal{P}_{\kappa f(X)}, \mathbf{Wp}_{X,0}\big)\right]$$

Proof. Recall the definitions of X and $\mathcal{X}'_{n-1}$ from the coupling construction in §SA.1. For any $R > 0$, define the event

$$E^n_X(R) = \left\{\tau_{X,r_n}\mathcal{X}'_{n-1} \cap B(X,R) = \mathcal{P}^n_{\kappa f(X)} \cap B(X,R)\right\}.$$

Recall $\mathbf{R}_n$ from Definition 4. For n sufficiently large and states of the world in the event $E^n_X(R) \cap \{R > \mathbf{R}_n\}$, by stabilization,

$$\begin{aligned}
\xi(X, \tau_{X,r_n}\mathcal{X}'_{n-1}, \mathbf{Wp}_{X,r_n}) &= \xi(X, \tau_{X,r_n}\mathcal{X}'_{n-1} \cap B(X,R), \mathbf{Wp}_{X,r_n}) \\
&= \xi(X, \mathcal{P}^n_{\kappa f(X)} \cap B(X,R), \mathbf{Wp}_{X,r_n}) \\
&= \xi(X, \mathcal{P}^n_{\kappa f(X)}, \mathbf{Wp}_{X,r_n}).
\end{aligned}$$

Therefore, for any $\epsilon > 0$ and n sufficiently large,

$$\mathbf{P}\left(\left|\xi(X, \tau_{X,r_n}\mathcal{X}'_{n-1}, \mathbf{W}\mathbf{p}_{X,r_n}) - \xi(X, \mathcal{P}^n_{\kappa f(X)}, \mathbf{W}\mathbf{p}_{X,r_n})\right| > \epsilon\right) \\ \leqslant \mathbf{P}(E^n_X(R)^c) + \mathbf{P}(\mathbf{R}_n > R), \quad \text{(SA.2.6)}$$

where $E^n_X(R)^c$ is the complement of $E^n_X(R)$. By stabilization,

$$\lim_{R\to\infty} \limsup_{n\to\infty} \mathbf{P}(\mathbf{R}_n > R) = 0.$$

By Lemma 3, for any $R > 0$,

$$\lim_{n\to\infty} \mathbf{P}(E^n_X(R)^c) = 0.$$

Hence, the left-hand side of (SA.2.6) tends to zero as $n \to \infty$.

By (SA.2.3) and the Vitali convergence theorem,

$$\left|\mathbf{E}[\xi(X, \tau_{X,r_n}\mathcal{X}'_{n-1}, \mathbf{W}\mathbf{p}_{X,r_n})] - \mathbf{E}[\xi(X, \mathcal{P}^n_{\kappa f(X)}, \mathbf{W}\mathbf{p}_{X,r_n})]\right| \to 0.$$

Since $\mathcal{P}^n_{\kappa f(X)} \overset{d}{=} \mathcal{P}_{\kappa f(X)}$ for any n, this implies

$$\mathbf{E}[\xi(X, \mathcal{P}^n_{\kappa f(X)}, \mathbf{W}\mathbf{p}_{X,r_n})] = \mathbf{E}[\xi(X, \mathcal{P}_{\kappa f(X)}, \mathbf{W}\mathbf{p}_{X,r_n})] = \mathbf{E}[\xi(X, \mathcal{P}_{\kappa f(X)}, \mathbf{W}^*\mathbf{p}_{X,r_n})].$$

Finally, since the right-hand side, viewed as a function of r_n, is assumed to be continuous at $r_n = 0$, it converges to $\mathbf{E}[\xi(X, \mathcal{P}_{\kappa f(X)}, \mathbf{W}^*\mathbf{p}_{X,0})]$. ■

By the coupling, the previous lemma shows that the expectation of the left-hand side of (SA.2.4) converges to the right-hand side. To prove Theorem 3, it suffices to show concentration of the variance. This requires a different coupling construction. Let X and Y be independently drawn from f, and let $\mathcal{P}_2$ be a Poisson process with intensity two on $\mathbb{R}^d \times [0, \infty)$, with $\mathcal{P}_2 \perp\!\!\!\perp X, Y$. Independently mark points of $\mathcal{P}_2$ "blue" with probability 0.5 and "gold" otherwise. Let $\mathcal{P}_1$ be the blue points and $\mathcal{Q}_1$ the gold points. Then by thinning, these are independent Poisson processes with unit intensity. The remainder of this construction follows Penrose and Yukich (2003).

- *Finite Model.* In the coupling construction in §SA.1, we construct $\mathcal{X}'_{n-1}$ from $\mathcal{P}_1$. Repeat this construction, except remove from $\mathcal{P}_{nf}$ a set of $\max\{N_n - (n-2), 0\}$ vectors chosen at random, and add $\max\{n - 2 - N_n, 0\}$ vectors drawn i.i.d. from f. Call the resulting set $\mathcal{X}'_{n-2}$.

- *X's Limit Model.* Let F_X be the half-space of points in $\mathbb{R}^d$ closer to X than to Y and

F_Y its complement. Let $\mathcal{P}^X_{nf(X)}$ be the restriction of $\mathcal{P}_1$ to $F_X \times [0, nf(X)]$ and $\mathcal{Q}^Y_{nf(X)}$ the restriction of $\mathcal{Q}_1$ to $F_Y \times [0, nf(X)]$. Let $\mathcal{P}^{n,+}_{\kappa f(X)}$ be the image of $\mathcal{P}^X_{nf(X)} \cup \mathcal{Q}^Y_{nf(X)}$ under $(w,t) \mapsto \tau_{X,r_n} w$ for $w \in \mathbb{R}^d$ and $t > 0$. Then $\mathcal{P}^{n,+}_{\kappa f(X)} \stackrel{d}{=} \mathcal{P}_{\kappa f(X)}$ for all n.

- *Y's Limit Model.* Analogously, let $\mathcal{P}^Y_{nf(Y)}$ be the restriction of $\mathcal{P}_1$ to $F_Y \times [0, nf(Y)]$ and $\mathcal{Q}^X_{nf(Y)}$ the restriction of $\mathcal{Q}_1$ to $F_X \times [0, nf(Y)]$. Let $\mathcal{P}^{n,+}_{\kappa f(Y)}$ be the image of $\mathcal{P}^Y_{nf(Y)} \cup \mathcal{Q}^X_{nf(Y)}$ under $(w,t) \mapsto \tau_{Y,r_n} w$. Then

$$\mathcal{P}^{n,+}_{\kappa f(X)} \perp\!\!\!\perp \mathcal{P}^{n,+}_{\kappa f(Y)} \tag{SA.2.7}$$

  for any n by the spatial independence property of Poisson processes.

By spatial independence and definition of $\mathbf{W}$,

$$\mathbf{W}(\mathbf{p}_{X,r_n}\mathcal{P}^{n,+}_{\kappa f(X)}) \perp\!\!\!\perp \mathbf{W}(\mathbf{p}_{Y,r_n}\mathcal{P}^{n,+}_{\kappa f(Y)}). \tag{SA.2.8}$$

Proof of Theorem 3. Given Lemma 4, it remains to show that the variance converges to zero. We follow the argument in the proof of Proposition 3.1 of Penrose and Yukich (2003). We have

$$\begin{aligned}
&\left|\xi(X, \tau_{X,r_n}(\mathcal{X}'_{n-2} \cup \{Y\}), \mathbf{W}\mathbf{p}_{X,r_n}) - \xi(X, \mathcal{P}^{n,+}_{\kappa f(X)}, \mathbf{W}\mathbf{p}_{X,r_n})\right| \xrightarrow{p} 0,\\
&\left|\xi(Y, \tau_{Y,r_n}(\mathcal{X}'_{n-2} \cup \{X\}), \mathbf{W}\mathbf{p}_{Y,r_n}) - \xi(Y, \mathcal{P}^{n,+}_{\kappa f(Y)}, \mathbf{W}\mathbf{p}_{Y,r_n})\right| \xrightarrow{p} 0.
\end{aligned}$$

This follows from arguments similar to the proof of Lemma 4 and the fact that, for any $R > 0$,

$$\lim_{n\to\infty} \mathbf{P}(B(X, Rr_n) \subset F_X) = \lim_{n\to\infty} \mathbf{P}(B(Y, Rr_n) \subset F_Y) = 1.$$

Thus, by uniform-square integrability and the Vitali convergence theorem,

$$\begin{aligned}
\Big|\mathbf{E}[\xi(X, \tau_{X,r_n}(\mathcal{X}'_{n-2} \cup \{Y\}, \mathbf{W}\mathbf{p}_{X,r_n})\\
\times \xi(Y, \tau_{Y,r_n}(\mathcal{X}'_{n-2} \cup \{X\}), \mathbf{W}\mathbf{p}_{Y,r_n})]\\
- \mathbf{E}[\xi(X, \mathcal{P}^{\kappa,+}_{\kappa f(X)}, \mathbf{W}\mathbf{p}_{X,r_n})\, \xi(Y, \mathcal{P}^{\kappa,+}_{\kappa f(Y)}, \mathbf{W}\mathbf{p}_{Y,r_n})]\Big| \to 0.
\end{aligned}$$

By (SA.2.7) and (SA.2.8), $\xi(X, \mathcal{P}^{\kappa,+}_{\kappa f(X)}, \mathbf{W}\mathbf{p}_{X,r_n}) \perp\!\!\!\perp \xi(Y, \mathcal{P}^{\kappa,+}_{\kappa f(Y)}, \mathbf{W}\mathbf{p}_{Y,r_n})$. Thus,

$$\begin{aligned}
\mathbf{E}\left[\xi(X, \mathcal{P}^{\kappa,+}_{\kappa f(X)}, \mathbf{W}\mathbf{p}_{X,r_n})\xi(Y, \mathcal{P}^{\kappa,+}_{\kappa f(Y)}, \mathbf{W}\mathbf{p}_{Y,r_n})\right]\\
= \mathbf{E}\left[\xi\big(X, \mathcal{P}_{\kappa f(X)}, \mathbf{W}^*\mathbf{p}_{X,r_n}\big)\right]^2 \to \mathbf{E}\left[\xi\big(X, \mathcal{P}_{\kappa f(X)}, \mathbf{W}^*\mathbf{p}_{X,0}\big)\right]^2,
\end{aligned} \tag{SA.2.9}$$

where convergence follows from the continuity assumption. Finally, by the coupling construction,

$$\mathbf{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\xi(X_i,\tau_{X_i,r_n}\mathcal{X}_n,\mathbf{Wp}_{X_i,r_n})\right)^2\right]$$
$$= n^{-1}\mathbf{E}\left[n^{-1}\xi(X,\tau_{X,r_n}(\mathcal{X}'_{n-2}\cup\{Y\}),\mathbf{Wp}_{X,r_n})^2\right]$$
$$+ (1-n^{-1})\mathbf{E}\left[\xi(X,\tau_{X,r_n}(\mathcal{X}'_{n-2}\cup\{Y\}),\mathbf{Wp}_{X,r_n})\right.$$
$$\left.\times\xi(Y,\tau_{Y,r_n}(\mathcal{X}'_{n-2}\cup\{X\}),\mathbf{Wp}_{Y,r_n})\right].$$

The first term on the right-hand side tends to zero by (SA.2.3), and the second term tends to (SA.2.9). Hence, the variance converges to zero by Lemma 4. ■

SA.3 Branching Processes

In this section, we show that the size of the relevant set J_i, defined in (18), is asymptotically bounded, which is the key step to verifying stabilization. The main result is Theorem 4, which defines the radius $\mathbf{R}_n$ in Definition 4 and proves it is asymptotically bounded. In order to prove $|J_i| = O_p(1)$, we need to show that $|C_j^+|$ and $|\mathcal{N}_{M(r_n)}(j,K)|$ are asymptotically bounded. We proceed by stochastically bounding these quantities by the sizes of certain branching processes and then establishing that these processes are subcritical, meaning that the total offspring generated by these processes is a.s. finite.

The first two lemmas are used to bound the sizes of $D(r_n)$-components. Key to this argument is the following branching process. Let $x' \in \mathbb{R}^d$, $z \in \mathbb{R}^{d_z}$, and $r \geqslant 0$. Let $\mathfrak{X}_r(x',z')$ denote the size of the following multi-type Galton-Walton branching process. The type space is $\mathbb{R}^d \times \mathbb{R}^{d_z}$, and the process begins at a particle of type (x',z'). Each type (x',z') particle is replaced in the next generation by offspring distributed according to a Poisson point process on $\mathbb{R}^d \times \mathbb{R}^{d_z}$ with intensity

$$\mathrm{d}\varphi_r(x',z';x'',z'') \equiv \kappa\bar{f}(1+r)p_1(x',z';x'',z'')\,\mathrm{d}\Phi^*(z'')\,\mathrm{d}x'', \tag{SA.3.1}$$

where Φ^* is defined in Assumption 4, p_1 is defined in (7), and $\bar{f} = \sup_{x\in\mathbb{R}^d} f(x)$. Offspring are independent across particles.

The offspring distribution of (x',z') can be equivalently represented as follows. Partition $\mathbb{R}^d$ into cubes with side length one centered at integer-valued elements of $\mathbb{R}^d$, and label the elements of the partition arbitrarily $1,2,\dots$. For each partition k, draw $N_k \sim \text{Poisson}(\kappa\bar{f}(1+$

$r)$), independently across partitions. Conditional on N_k, draw positions $X_1, \dots, X_{N_k}$ i.i.d. and uniformly distributed on the partition. Draw attributes $\{Z_i : i = 1, \dots, N_k\} \overset{iid}{\sim} \Phi^*$ conditional on N_k and random shocks $\{\zeta_{ij} : i, j = 1, \dots, N_k\}$ i.i.d., independently across partitions. Let (X_i, Z_i) be an offspring of (x', z') with probability $p_1(x', z'; X_i, Z_i)$ for all $i = 1, \dots, N_k$ and partitions k. For intuition on this, observe that if, in the expression of (SA.3.1), we replace $\bar{f}$ with $f(x)$ and $\Phi^*(z'')$ with $\Phi(z'' \,|\, \mathbf{p}_{x,r_n}(x''))$ and set $r = 1$, then the resulting number of offspring has the same distribution as the number of non-robust links formed by a node with type (x', z') under the limit model

$$(V, \lambda, \mathcal{P}_{\kappa f(x)} \cup \{x\}, \mathbf{W}\mathbf{p}_{x,r_n}, 1). \tag{SA.3.2}$$

The probability that the branching process started by (x', z') survives indefinitely is given by

$$\rho_r(x', z') = \mathbf{P}(\mathfrak{X}_r(x', z') = \infty).$$

Let T_r be the functional satisfying

$$(T_r \circ h)(x', z') = \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d_z}} h(x'', z'')\, \mathrm{d}\varphi_r(x', z'; x'', z'').$$

Our first result shows that the survival probability ρ_r is the solution of a fixed-point equation. All proofs can be found at the end of this section.

Lemma 5. *ρ_r satisfies*

$$\rho_r = 1 - exp\left\{-(T_r \circ \rho_r)\right\}. \tag{SA.3.3}$$

Our next result establishes that the branching process is subcritical. Recall the definition of $Z(x)$ from §SA.1.

Lemma 6. *Under Assumption 6, $\rho_r(x, Z(x)) = 0$ a.s. for any $x \in \mathbb{R}^d$ and r sufficiently small.*

We next study a fixed-depth branching process used to bound the sizes of K-neighborhoods in $M(r_n)$. Recall the definition of $\tilde{p}_r(\cdot)$ from (5). Let $\tilde{\mathfrak{X}}_r(x', z'; K)$ denote the size of the branching process on $\mathbb{R}^{d+d_z}$ after K generations, starting at a particle of type (x', z'), where the offspring distribution of (x', z') is given by a Poisson point process on $\mathbb{R}^{d+d_z}$ with intensity

$$\mathrm{d}\tilde{\varphi}_r(x', z'; x'', z'') \equiv \kappa \bar{f}(1 + r)\tilde{p}_1(x', z'; x'', z'')\, \mathrm{d}\Phi^*(z'')\, \mathrm{d}x''. \tag{SA.3.4}$$

Let $\tilde{\rho}_r(x', z'; K) = \mathbf{P}(\tilde{\mathfrak{X}}_r(x', z'; K) = \infty)$. The next lemma is the analog of Lemma 6 for the fixed depth process.

Lemma 7. *Under Assumption 4, $\tilde{\rho}_r(x, Z(x); K) = 0$ a.s. for any $x \in \mathbb{R}^d$ and r sufficiently small.*

For an initial particle (x', z'), let $\mathbf{B}_r(x', z'; K)$ be the set of particles (x'', z'') in a fixed-depth branching process for K generations with intensity given by (SA.3.4). Note that $|\mathbf{B}_r(x', z'; K)| = \tilde{\mathfrak{X}}_r(x', z'; K)$. Since the latter is a.s. finite by the previous lemma, we naturally expect that, due to homophily, the largest distance between x' and any other particle is also a.s. finite. We prove this in the next lemma.

Lemma 8. *Under Assumptions 4 and 8(a), for any $x, x' \in \mathbb{R}^d$, and $K > 0$,*

$$\mathbf{P}\left(\max_{(x'',z'') \in \mathbf{B}_\kappa(x', Z(x'); K)} ||x' - x''|| = \infty \,\middle|\, Z(x') \right) = 0.$$

Recall from §SA.1 the definition of X. Let $J(X, \tau_{X,r_n}\mathcal{X}_n, \mathbf{W}\mathbf{p}_{X,r_n}, 1) = J(X, \mathcal{X}_n, \mathbf{W}, r_n)$ (see (SA.1.1)) be the relevant set (18) of the node positioned at X under the finite model and $J(X, \mathcal{P}_{\kappa f(X)} \cup \{X\}, \mathbf{W}\mathbf{p}_{X,r_n}, 1)$ the relevant set of X under the limit model, where we label nodes, say, by lexicographic order of their positions. The next lemma shows that the size of the relevant set is stochastically bounded by the following branching process. For an initial particle (x', z'), first generate $\mathbf{B}_{r_n}(x', z'; K)$, the set of particles in a fixed-depth branching process for K generations with intensity given by (SA.3.4). This is our branching-process approximation of the K-neighborhood of a node in $M(1)$. Then to approximate $D(1)$-component sizes, for each particle $(x'', z'') \in \mathbf{B}_{r_n}(x', z'; K)$, initiate independent branching processes with intensities given by (SA.3.1) whose sizes consequently have the same distribution as $\mathfrak{X}_{r_n}(x'', z'')$. Lastly, for each particle generated by the latter process, initiate a fixed-depth branching process for 1 generation with intensity given by (SA.3.4). Let $\hat{\mathfrak{X}}_{r_n}(x', z'; K)$ denote the size of the overall process.

Lemma 9. *For any $\epsilon > 0$ and n sufficiently large,*

$$\begin{aligned} \max\big\{&\mathbf{P}(|J(X, \mathcal{X}_n, \mathbf{W}, r_n)| > \epsilon), \\ &\mathbf{P}(|J(X, \mathcal{P}_{\kappa f(X)} \cup \{X\}, \mathbf{W}\mathbf{p}_{X,r_n}, 1)| > \epsilon)\big\} \\ &\leqslant \mathbf{P}(\hat{\mathfrak{X}}_{r_n}(X, Z(X); K) > \epsilon). \end{aligned}$$

We now state the main result of this section. Let $\ell(X')$ be the node label associated with $X' \in \mathcal{X}_n$ or $X' \in \mathcal{P}_{\kappa f(X)} \cup \{X\}$ (recall labels are given by $\mathcal{N}_n$ for the former and by lexicographic order of positions for the latter). Define

$$\mathbf{R}_n(X) = \max\Bigg\{ \max_{X' \in \mathcal{X}'_{n-1} \colon \ell(X') \in J(X, \mathcal{X}_n, \mathbf{W}, r_n)} r_n^{-1} ||X - X'||, \\ \max_{X' \in \mathcal{P}_{\kappa f(X)} \colon \ell(X') \in J(X, \mathcal{P}_{\kappa f(X)} \cup \{X\}, \mathbf{W}\mathbf{p}_{X, r_n}, 1)} ||X - X'|| \Bigg\}. \quad \text{(SA.3.5)}$$

By construction, $B(X, \mathbf{R}_n(X))$ contains all nodes in the relevant set under either the finite or limit model. Formally, under the finite model (Assumption 3), $B(X, \mathbf{R}_n(X))$ contains $\{X' \in \tau_{X,r_n}\mathcal{X}_n : \ell(\mathbf{p}_{X,r_n}X') \in J(X, \mathcal{X}_n, \mathbf{W}, r_n)\}$, and under the limit model (SA.3.2) with $x = X$, it contains $\{X' \in \mathcal{P}_{\kappa f(X)} \cup \{X\} : \ell(X') \in J(X, \mathcal{P}_{\kappa f(X)} \cup \{X\}, \mathbf{W}\mathbf{p}_{X,r_n}, 1)\}$. The following theorem show that this radius is asymptotically bounded.

Theorem 4. *Under Assumptions 4, 6, and 8(a),* $\mathbf{R}_n(X) = O_p(1)$.

Proof of Lemma 5. This is a standard branching-process argument. Let μ_k denote the probability law associated with the uniform distribution on partition k. Fix the starting particle of the process (x', z').

Recall the representation of the offspring distribution at the start of this section. In the kth partition of $\mathbb{R}^d$, label the offspring lying in that partition $k1, k2, \dots, kN_k$, where N_k is the total offspring in the partition. Let (X_{kj}, Z_{kj}) be the type of offspring kj. We claim

$$\begin{aligned}
\rho_r(x', z') &= 1 - \mathbf{E}\left[\prod_{k=1}^{\infty}\prod_{j=1}^{N_k}(1 - \rho_r(X_{kj}, Z_{kj})p_1(x', z'; X_{kj}, Z_{kj}))\right] \\
&= 1 - \prod_{k=0}^{\infty}\mathbf{E}\left[\mathbf{E}\left[(1 - \rho_r(X_{kj}, Z_{kj})p_1(x', z'; X_{kj}, Z_{kj})) \,\middle|\, N_k\right]^{N_k}\right] \\
&= 1 - \prod_{k=0}^{\infty}\mathbf{E}\left[\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d_z}} (1 - \rho_r(x'', z'')p_1(x', z'; x'', z''))\, \mathrm{d}\Phi^*(z'')\, \mathrm{d}\mu_k(x'')\right)^{N_k}\right].
\end{aligned}$$

The second line uses the assumption that types are i.i.d. given N_k and monotone convergence. Then using the fact that $\mathbf{E}[w^{\text{Poisson}(\lambda)}] = e^{\lambda(w-1)}$ (e.g. Bollobás et al., 2007, proof of Theorem

12.5), the third line equals one minus

$$\prod_{k=0}^{\infty} \exp\left\{\kappa \bar{f}(1+r)\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d_z}} \rho_r(x'',z'')p_1(x',z';x'',z'')\,\mathrm{d}\Phi^*(z'')\,\mathrm{d}\mu_k(x'')\right\}$$
$$= \exp\left\{-\kappa \bar{f}(1+r)\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d_z}} \rho_r(x'',z'')p_1(x',z';x'',z'')\,\mathrm{d}\Phi^*(z'')\mathrm{d}x''\right\}$$

The right-hand side uses monotone convergence. ■

Proof of Lemma 6. For any $h : \mathbb{R}^d \times \mathbb{R}^{d_z} \to \mathbb{R}$, let

$$||h|| = \sup_{x' \in \mathbb{R}^d} \left[\int_{\mathbb{R}^{d_z}} h(x',z')^2\,\mathrm{d}\Phi^*(z')\right]^{1/2}.$$

Let $\mathcal{H}$ be the set of functions $h : \mathbb{R}^d \times \mathbb{R}^{d_z} \to [0,1]$ for which $||h|| \leqslant 1$. Observe that T_r is a linear map on $\mathcal{H}$. We further claim that T_r is a bounded map. In particular, by the Cauchy-Schwarz inequality, for any $h \in \mathcal{H}$, $||T_r h||$ is bounded above by

$$||h||\ \kappa \bar{f}\,(1+r) \sup_{x' \in \mathbb{R}^d} \left(\int_{\mathbb{R}^{d_z}} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^{d_z}} p_1(x',z';x'',z'')^2 \times \mathrm{d}\Phi^*(z'')\right)^{1/2} \mathrm{d}x''\right)^2 \mathrm{d}\Phi^*(z')\right)^{1/2}. \quad \text{(SA.3.6)}$$

The term multiplying $||h||$ is bounded above by a finite constant by (8). Since T_r is a bounded linear map, the operator norm of T_r is given by $||T_r|| = \sup\{||T_r h|| : h \in \mathcal{H}\}$. This implies that for any $h \in \mathcal{H}$,

$$||T_r h|| \leqslant ||T_r||\,||h||. \quad \text{(SA.3.7)}$$

Now, let h be a solution of (SA.3.3) and suppose, to get a contradiction, that $h(x,z) \neq 0$ for some $\tilde{x} \in \mathbb{R}^d$ and set S' of $z \in \mathbb{R}^{d_z}$ with positive probability measure under the conditional probability law of $\Phi(\cdot \mid \tilde{x})$. Let $S = \{\tilde{x}\} \times S'$. Since h is a solution to (SA.3.3), $T_r h \geqslant h$, where the inequality is strict on S (see e.g. Bollobás et al., 2007, Lemma 5.8(ii)). Then it follows that

$$||T_r h|| > ||h||. \quad \text{(SA.3.8)}$$

On the other hand, suppose for the moment that

$$||T_r 1|| \leqslant 1. \quad \text{(SA.3.9)}$$

Then $h \in \mathcal{H}$, since $h \in [0,1]$ and

$$||h|| \leqslant ||T_r h|| \leqslant ||T_r 1|| \leqslant 1,$$

which implies h satisfies (SA.3.7). This contradicts (SA.3.8), since $||T_r|| \leqslant ||T_r 1|| \leqslant 1$. Thus, any solution h of (SA.3.3) satisfies $h(x, Z(x)) = 0$ a.s. for any $x \in \mathbb{R}^d$. In particular, this holds for ρ_r, since by Lemma 5, ρ_r is a solution of (SA.3.3).

It remains to prove (SA.3.9) holds for sufficiently small r. Note that $||T_r 1||$ equals

$$\kappa \bar{f}\,(1+r) \sup_{x' \in \mathbb{R}^d} \underbrace{\left(\int_{\mathbb{R}^{d_z}} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^{d_z}} p_1(x', z'; x'', z'')\, \mathrm{d}\Phi^*(z'')\, \mathrm{d}x'' \right)^2 \mathrm{d}\Phi^*(z') \right)^{1/2}}_{\beta(x')}. \tag{SA.3.10}$$

By (9), for any $x' \in \mathbb{R}^d$, $\kappa \bar{f} \beta(x') < 1$. Hence, for all r sufficiently small, $\kappa \bar{f}(1+r)\beta(x') < 1$, which implies $\kappa \bar{f}(1+r)\sup_{x'} \beta(x') \leqslant 1$ for such r. By (SA.3.10), this is precisely (SA.3.9). ■

Proof of Lemma 7. Following the argument in the proof of Lemma 5, $\tilde{\rho}_r(x', z'; K)$ equals

$$1 - \exp\left\{ -\kappa \bar{f}\,(1+r) \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d_z}} \tilde{\rho}_r(x'', z''; K-1) \tilde{p}_1(x', z'; x'', z'')\, \mathrm{d}\Phi^*(z'')\, \mathrm{d}x'' \right\}.$$

This is less than or equal to

$$\kappa \bar{f}\,(1+r) \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d_z}} \tilde{\rho}_r(x'', z''; K-1) \tilde{p}_1(x', z'; x'', z'')\, \mathrm{d}\Phi^*(z'')\, \mathrm{d}x''.$$

After zero generations, there is exactly one particle, so by construction, $\tilde{\rho}_r(x', z'; 0) = 0$. Then by the previous two equations, $\tilde{\rho}_r(x', z'; 1) = 0$, provided

$$\kappa \bar{f}\,(1+r) \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d_z}} \tilde{p}_1(x', z'; x'', z'')\, \mathrm{d}\Phi^*(z'')\, \mathrm{d}x'' < \infty. \tag{SA.3.11}$$

The left-hand side of (SA.3.11) is bounded above by

$$\begin{aligned} &\kappa \bar{f}\,(1+r) \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d_z}} \tilde{p}_1(x', z'; x'', z'')\, \mathrm{d}\Phi^*(z'')\, \mathrm{d}x'' \\ &\qquad \to \kappa \bar{f} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d_z}} \tilde{p}_1(x', z'; x'', z'')\, \mathrm{d}\Phi^*(z'')\, \mathrm{d}x'' \end{aligned} \tag{SA.3.12}$$

as $r \to 0$. By Assumption 4, this is finite a.s. for any $x' \in \mathbb{R}^d$ and $z' \stackrel{d}{=} Z(x')$.

Hence, for r sufficiently small, (SA.3.11) holds, from which it follows that $\tilde{\rho}_r(x', z'; 1) = 0$, as desired. Repeating this argument, $\tilde{\rho}_r(x', z'; K) = 0$ for arbitrary K. ∎

Proof of Lemma 8. Fix $x, x' \in \mathbb{R}^d$, z' a draw from $\Phi(\cdot \mid x')$, and $K > 0$. Define

$$\hat{\rho}_\kappa(x', z'; K) \equiv \mathbf{P}\left(\max_{(x'', z'') \in \mathbf{B}_\kappa(x', z'; K)} ||x' - x''|| = \infty \right).$$

For the maximum distance to be infinite, it must be either that (x', z') has some offspring (x'', z'') for which either $||x' - x''|| = \infty$, or $\max_{(x''', z''') \in \mathbf{B}_\kappa(x'', z''; K-1)} ||x'' - x'''|| = \infty$. Thus, following the argument in the proof of Lemma 7, $\hat{\rho}_\kappa(x', z'; K)$ is bounded above by

$$\kappa \bar{f}\,(1 + \kappa) \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d_z}} \hat{\rho}_\kappa(x'', z''; K - 1) \tilde{p}_1(x', z'; x'', z'')\, \mathrm{d}\Phi^*(z'')\, \mathrm{d}x'' + \hat{\rho}_\kappa(x', z'; 1). \quad \text{(SA.3.13)}$$

Consider the second term. We have

$$\begin{aligned} \mathbf{E}\left[\max_{(x'', z'') \in \mathbf{B}_\kappa(x', z'; 1)} ||x' - x''|| \right] &\leqslant \mathbf{E}\left[\sum_{(x'', z'') \in \mathbf{B}_\kappa(x', z'; 1)} ||x' - x''|| \right] \\ &= \kappa \bar{f}\,(1 + \kappa) \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d_z}} \tilde{p}_1(x', z'; x'', z'') ||x' - x''||\, \mathrm{d}\Phi^*(z'')\, \mathrm{d}x''. \end{aligned}$$

By Assumption 4, the right-hand side is a.s. finite for any $x' \in \mathbb{R}^d$ and $z' \overset{d}{=} Z(x')$. Thus, $\hat{\rho}_\kappa(x', z'; 1) = 0$.

Now consider the first term in (SA.3.13). After zero generations, there is exactly one particle, so by construction, $\hat{\rho}_\kappa(x'', z''; 0) = 0$. Then since for $K = 1$ (SA.3.13) is an upper bound on $\hat{\rho}_\kappa(x', z'; 1)$, the latter probability is zero if

$$\kappa \bar{f}\,(1 + \kappa) \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d_z}} \tilde{p}_1(x', z'; x'', z'')\, \mathrm{d}\Phi^*(z'')\, \mathrm{d}x'' < \infty.$$

This holds by Assumption 4. Repeating this argument, we have $\hat{\rho}_\kappa(x', z'; K) = 0$ for arbitrary K. ∎

Proof of Lemma 9. By definition, for $i = \ell(X)$, $J(X, \mathcal{X}_n, \mathbf{W}, r_n)$ is constructed by branching out to $\mathcal{N}_{M(r_n)}(i, K)$, then branching out to C_j for each $j \in \mathcal{N}_{M(r_n)}(i, K)$, then branching out to the set of nodes ℓ that are robustly linked to each $k \in C_j$. By comparison, the process $\hat{\mathfrak{X}}_{r_n}(X_i, Z(X_i); K)$ replaces each of the sets in each step with a dominating branching process. More specifically, it replaces each $\mathcal{N}_{M(r_n)}(i,K)$ with a fixed-depth branching process with intensity (SA.3.4) and depth K. It also replaces each C_j with a branching process with intensity (SA.3.1). The only discrepancy is the last step, which can be explained as follows. The set of nodes ℓ robustly connected to any $k \in C_j$ is contained in $\mathcal{N}_{M(r_n)}(k,K)$. The process $\hat{\mathfrak{X}}_{r_n}(X_i, Z(X_i); K)$ replaces the latter set with a fixed-depth branching process with intensity (SA.3.4) and depth 1.

Let $C(X, \mathcal{X}_n, \mathbf{W}, D(r_n))$ be the $D(r_n)$-component of the node positioned at X under the finite model $(V, \lambda, \mathcal{X}_n, \mathbf{W}, r_n)$. Let $C(X, \mathcal{P}_{\kappa f(X)} \cup \{X\}, \mathbf{W}\mathbf{p}_{X,r_n}, D(1))$ be its $D(1)$-component under the limit model (SA.3.2) with $x = X$. Define

$$\mathcal{N}_K(X, \mathcal{X}_n, \mathbf{W}, M(r_n)) \quad \text{and} \quad \mathcal{N}_K(X, \mathcal{P}_{\kappa f(X)} \cup \{X\}, \mathbf{W}\mathbf{p}_{X,r_n}, M(1))$$

respectively as the K-neighborhood in the network $M(r_n)$ of the node positioned at X under the finite model and the K-neighborhood in $M(1)$ under the limit model.

We will show that $|C(X, \mathcal{X}_n, \mathbf{W}, D(r_n))|$ and $|C(X, \mathcal{P}_{\kappa f(X)} \mathbf{W}\mathbf{p}_{X,r_n}, D(1))|$ are both stochastically dominated by $\mathfrak{X}_{r_n}(X, Z(X))$. A similar argument shows that

$$\max\left\{|\mathcal{N}_K(X, \mathcal{X}_n, \mathbf{W}, M(r_n))|, |\mathcal{N}_K(X, \mathcal{P}_{\kappa f(X)} \mathbf{W}\mathbf{p}_{X,r_n}, M(1))|\right\}$$

is stochastically dominated by $\tilde{\mathfrak{X}}_{r_n}(X, Z(X); K)$. The conclusion of the lemma then follows from the discussion in the first paragraph of this proof.

Condition on $(X, Z(X)) = (x, z)$. We first establish stochastic dominance under the finite model. We can estimate the size of node X's $D(r_n)$-component using a standard breadth-first search argument on $D(r_n)$ starting at node X (see e.g. Bollobás et al., 2007, Lemma 9.6). That is, we branch out to X's 1-neighborhood in $D(r_n)$ (her "offspring"), recording the number of neighbors as $B_1(n)$. Then we successively branch out to the 1-neighborhood of one of her neighbors, *not including nodes previously visited*, to obtain recording the total newly explored neighbors as $B_2(n)$. We repeat this process indefinitely. Then $\sum_{m=1}^{\infty} B_m(n) + 1 = |C(X, \mathcal{X}_n, \mathbf{W}, D(r_n))|$.

At each step m of the breadth-first search, we explore the 1-neighborhood of a node of type (x', z'). Conditional on this type, the number of newly explored nodes $B_m(n)$ is stochastically dominated by $B'_m(n) \sim \text{Binomial}(n-1, p_{r_n}(x', z'))$, where

$$\begin{aligned} p_{r_n}(x', z') &= \mathbf{E}[p_{r_n}(x', z'; X, Z(X))] \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d_z}} p_1(x', z'; x'', z'') \Phi(z'' \mid \mathbf{p}_{x,r_n}(x'')) r_n^d f(\mathbf{p}_{x,r_n}(x'')) \, \mathrm{d}x'', \end{aligned} \quad \text{(SA.3.14)}$$

since this is the distribution of the number of neighbors. The right-hand side of (SA.3.14) uses a change of variables $(x', x'') \mapsto (\tau_{x,r_n} x', \tau_{x,r_n} x'')$.

Thus, $\sum_{m=1}^{\infty} B_m(n)$ is stochastically dominated by $\sum_{m=1}^{\infty} B'_m(n)$. The latter is the size of a multi-type branching process started at (x, z) in which we replace each particle of type (x', z') in the next generation with a set of particles with i.i.d. types in $\mathbb{R}^d \times \mathbb{R}^{d_z}$, and the number of such particles is distributed Binomial$(n-1, p_{r_n}(x', z'))$.

It is a fact that, for n sufficiently large, a Binomial$(n-1, p_{r_n}(x', z'))$ random variable is stochastically dominated by a Poisson$(n(1+r_n)p_{r_n}(x', z'))$ random variable (see e.g. Bollobás et al., 2007, Theorem 12.5). By definition, $\mathfrak{X}_{r_n}(x, z)$ is the size of a multi-type branching process started at (x, z) with conditional offspring distribution Poisson$(n(1+r_n)\bar{p}_{r_n}(x', z'))$, where

$$\bar{p}_{r_n}(x', z') = \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d_z}} p_1(x', z'; x'', z'') \Phi^*(z'') r_n^d \bar{f} \, \mathrm{d}x'' \geqslant p_{r_n}(x', z').$$

It follows that for n sufficiently large, $\sum_{m=1}^{\infty} B'_m(n)$ is stochastically dominated by $\mathfrak{X}_{r_n}(x, z)$, as desired.

This establishes the claim for the finite model. For the limit model (SA.3.2), the offspring distribution is exactly Poisson, so the size of the breadth-first search can be directly bounded by $\mathfrak{X}_{r_n}(x, z)$. ∎

Proof of Theorem 4. We will show that the first element in the outermost maximum in the definition of $\mathbf{R}_n(X)$ is $O_p(1)$. The argument for the second element is similar. Let $J_0 = J(X, \mathcal{X}_n, \mathbf{W}, r_n)$. By the law of total probability,

$$\mathbf{P}\left(\max_{j \in J_0} r_n^{-1} ||X - X_j|| > C \right) \leqslant \mathbf{P}\left(|J_0| > B\right) + \mathbf{P}\left(\max_{j \in J_0} r_n^{-1} ||X - X_j|| > C \cap |J_0| \leqslant B \right). \quad \text{(SA.3.15)}$$

We will consider each of the two terms on the right-hand side in turn.

Step 1. Consider $\mathbf{P}(|J_0| > B)$. By Lemma 9, $|J_0|$ is stochastically dominated by $\hat{\mathfrak{X}}_{r_n}(X, Z(X); K)$. Recall that the latter is generated first by running a fixed-depth branching process $\mathbf{B}_{r_n}(X, Z(X); K)$; second, generating $\{\mathfrak{X}_{r_n}(x', z') : (x', z') \in \mathbf{B}_{r_n}(X, Z(X); K)\}$ independently conditional on $\mathbf{B}_{r_n}(X, Z(X); K)$; and third, conditional on the set of nodes Ψ_n generated in step 2, drawing $\{\tilde{\mathfrak{X}}_{r_n}(x'', z''; 1) : (x'', z'') \in \Psi_n\}$ independently.

We want to show that $\hat{\mathfrak{X}}_{r_n}(X, Z(X); K) = O_p(1)$. This requires some new definitions. Take any $\mathfrak{X}_{r_n}(x', z')$ generated in the second step and add all particles generated by this process to a set $\mathcal{T}_{r_n}(x', z')$. (So the size of $\mathcal{T}_{r_n}(x', z')$ equals $\mathfrak{X}_{r_n}(x', z')$.) Define

$$\mathfrak{X}^{23}_{r_n}(x', z') = |\mathcal{T}_{r_n}(x', z')| + \sum_{(x'', z'') \in \mathcal{T}_{r_n}(x', z')} \tilde{\mathfrak{X}}_{r_n}(x'', z''; 1),$$

That is, we add to the size of $\mathcal{T}_{r_n}(x', z')$ the size of each 1-depth branching process generated in step 3 that is initialized at a particle in $\mathcal{T}_{r_n}(x', z')$. Then

$$\begin{aligned} \mathbf{P}\left(\hat{\mathfrak{X}}_{r_n}(X, Z(X); K) > B\right) &= \mathbf{P}\left(\sum_{(x', z') \in \mathbf{B}_{r_n}(X, Z(X); K)} \mathfrak{X}^{23}_{r_n}(x', z') > B\right) \\ &\leqslant \mathbf{P}\left(\sum_{(x', z') \in \mathbf{B}_{r_n}(X, Z(X); K)} \mathfrak{X}^{23}_{r_n}(x', z') > B \cap |\mathbf{B}_{r_n}(X, Z(X); K)| \leqslant B'\right) \\ &\qquad + \mathbf{P}\left(|\mathbf{B}_{r_n}(X, Z(X); K)| > B'\right). \end{aligned}$$

Since $|\mathbf{B}_{r_n}(X, Z(X); K)| \overset{d}{=} \tilde{\mathfrak{X}}_{r_n}(X, Z(X); K)$, by Lemma 7, for any $\varepsilon > 0$, we can choose B' large enough such that

$$\limsup_{n \to \infty} \mathbf{P}\left(|\mathbf{B}_{r_n}(X, Z(X); K)| > B'\right) < \varepsilon/4.$$

On the other hand,

$$\begin{aligned} &\mathbf{P}\Bigg(\sum_{(x', z') \in \mathbf{B}_{r_n}(X, Z(X); K)} \mathfrak{X}^{23}_{r_n}(x', z') > B \cap |\mathbf{B}_{r_n}(X, Z(X); K)| \leqslant B'\Bigg) \\ &\quad = \mathbf{E}\Bigg[\mathbf{P}\Bigg(\sum_{(x', z') \in \mathbf{B}_{r_n}(X, Z(X); K)} \mathfrak{X}^{23}_{r_n}(x', z') > B \,\Bigg|\, \mathbf{B}_{r_n}(X, Z(X); K)\Bigg) \\ &\qquad\qquad \times \mathbf{1}\{|\mathbf{B}_{r_n}(X, Z(X); K)| \leqslant B'\}\Bigg] \\ &\quad \leqslant \mathbf{E}\Bigg[\sum_{(x', z') \in \mathbf{B}_{r_n}(X, Z(X); K)} \mathbf{P}\left(\mathfrak{X}^{23}_{r_n}(x', z') > B/B' \,\middle|\, \mathbf{B}_{r_n}(X, Z(X); K)\right) \\ &\qquad\qquad \times \mathbf{1}\{|\mathbf{B}_{r_n}(X, Z(X); K)| \leqslant B'\}\Bigg]. \end{aligned}$$

Since $\mathfrak{X}^{23}_{r_n}(x', z') \perp\!\!\!\perp \mathbf{B}_{r_n}(X, Z(X); K) \,\big|\, (x', z')$, for n sufficiently large, the right-hand side is bounded above by

$$B'\mathbf{E}\left[\max_{(x',z')\in\mathbf{B}_{r_n}(X,Z(X);K)}\mathbf{P}\left(\mathfrak{X}^{23}_{r_n}(x',z')>B/B'\,\middle|\,x',z'\right)\mathbf{1}\{|\mathbf{B}_{r_n}(X,Z(X);K)|\leqslant B'\}\right]. \tag{SA.3.16}$$

Note that for n sufficiently large, $|\mathbf{B}_{r_n}(X,Z(X);K)|\overset{d}{=}\tilde{\mathfrak{X}}_{r_n}(X,Z(X);K)$ is stochastically dominated by $\tilde{\mathfrak{X}}^{23}_{r'}(X,Z(X);K)$ for some $r'\in(0,\kappa]$ by inspection of the intensity measure (SA.3.4). Hence,

$$\text{(SA.3.16)}\leqslant B'\mathbf{E}\left[\max_{(x',z')\in\mathbf{B}_{r'}(X,Z(X);K)}\mathbf{P}\left(\mathfrak{X}^{23}_{r_n}(x',z')>B/B'\,\middle|\,x',z'\right)\mathbf{1}\{|\mathbf{B}_{r'}(X,Z(X);K)|\leqslant B'\}\right]. \tag{SA.3.17}$$

Now, assume for the moment that

$$\lim_{B\to\infty}\limsup_{n\to\infty}\mathbf{P}\left(\mathfrak{X}^{23}_{r_n}(x,Z(x))>B\,\middle|\,Z(x)\right)=0\quad\text{a.s. for any } x\in\mathbb{R}^d \text{ and } r' \text{ sufficiently small.} \tag{SA.3.18}$$

(We will prove this claim in step 2 below.) Then

$$\lim_{B\to\infty}\limsup_{n\to\infty}\max_{(x',z')\in\mathbf{B}_{r'}(X,Z(X);K)}\mathbf{P}\left(\mathfrak{X}^{23}_{r_n}(x',z')>B/B'\,\middle|\,x',z'\right)$$
$$\times\,\mathbf{1}\{|\mathbf{B}_{r'}(X,Z(X);K)|\leqslant B'\}\overset{a.s.}{=}0.$$

Hence, by the bounded convergence theorem, for any $\varepsilon, B'>0$, we can choose B large enough such that

$$\limsup_{n\to\infty}\text{(SA.3.16)}\leqslant\limsup_{n\to\infty}\text{(SA.3.17)}<\varepsilon/4.$$

We have therefore shown that for any $\varepsilon>0$, we can choose B large enough such that

$$\limsup_{n\to\infty}\mathbf{P}\left(|J_0|>B\right)<\varepsilon/2. \tag{SA.3.19}$$

Step 2. We prove (SA.3.18). The argument is the same as step 1. We have

$$\mathbf{P}\left(\mathfrak{X}^{23}_{r_n}(x,Z(x))>B\right)=\mathbf{P}\left(\sum_{(x',z')\in\mathcal{T}_{r_n}(x,Z(x))}\tilde{\mathfrak{X}}_{r_n}(x',z';1)>B\right)$$
$$\leqslant\mathbf{P}\left(\sum_{(x',z')\in\mathcal{T}_{r_n}(x,Z(x))}\tilde{\mathfrak{X}}_{r_n}(x',z';1)>B\cap|\mathcal{T}_{r_n}(x,Z(x))|\leqslant B'\right)$$
$$+\,\mathbf{P}\left(|\mathcal{T}_{r_n}(x,Z(x))|>B'\right).$$

Since $|\mathcal{T}_{r_n}(x, Z(x))| = \mathfrak{X}_{r_n}(x, Z(x))$, by Lemma 6, for any $\varepsilon > 0$, we can choose B' large enough such that

$$\limsup_{n\to\infty} \mathbf{P}\left(|\mathcal{T}_{r_n}(x, Z(x))| > B'\right) < \varepsilon/2. \tag{SA.3.20}$$

On the other hand,

$$\begin{aligned}
&\mathbf{P}\Bigg(\sum_{(x',z')\in\mathcal{T}_{r_n}(x,Z(x))} \tilde{\mathfrak{X}}_{r_n}(x', z'; 1) > B \cap |\mathcal{T}_{r_n}(x, Z(x))| \leqslant B'\Bigg) \\
&\qquad = \mathbf{E}\Bigg[\mathbf{P}\Bigg(\sum_{(x',z')\in\mathcal{T}_{r_n}(x,Z(x))} \tilde{\mathfrak{X}}_{r_n}(x', z'; 1) > B \,\Bigg|\, \mathcal{T}_{r_n}(x, Z(x))\Bigg) \\
&\qquad\qquad \times \mathbf{1}\{|\mathcal{T}_{r_n}(x, Z(x))| \leqslant B'\}\Bigg] \\
&\qquad \leqslant \mathbf{E}\Bigg[\sum_{(x',z')\in\mathcal{T}_{r_n}(x,Z(x))} \mathbf{P}\left(\tilde{\mathfrak{X}}_{r_n}(x', z'; 1) > B/B' \,\middle|\, \mathcal{T}_{r_n}(x, Z(x))\right) \\
&\qquad\qquad \times \mathbf{1}\{|\mathcal{T}_{r_n}(x, Z(x))| \leqslant B'\}\Bigg].
\end{aligned}$$

Since $\tilde{\mathfrak{X}}_{r_n}(x', z'; 1) \perp\!\!\!\perp \mathcal{T}_{r_n}(x, Z(x)) \,|\, (x', z')$, for n sufficiently large, the right-hand side is bounded above by

$$B'\mathbf{E}\left[\max_{(x',z')\in\mathcal{T}_{r_n}(x,Z(x))} \mathbf{P}\left(\tilde{\mathfrak{X}}_{r_n}(x', z'; 1) > B/B' \,\middle|\, x', z'\right) \mathbf{1}\{|\mathcal{T}_{r_n}(x, Z(x))| \leqslant B'\}\right]. \tag{SA.3.21}$$

Note that for n sufficiently large, $|\mathcal{T}_{r_n}(x, Z(x))| \overset{d}{=} \mathfrak{X}_{r_n}(x, Z(x))$ is stochastically dominated by $\mathfrak{X}_{r'}(x, Z(x))$ for some $r' \in (0, \kappa]$ by inspection of the intensity measure (SA.3.1). Hence,

$$\text{(SA.3.21)} \leqslant B'\mathbf{E}\left[\max_{(x',z')\in\mathcal{T}_{r'}(x,Z(x))} \mathbf{P}\left(\tilde{\mathfrak{X}}_{r_n}(x', z'; 1) > B/B' \,\middle|\, x', z'\right) \mathbf{1}\{|\mathcal{T}_{r'}(x, Z(x))| \leqslant B'\}\right]. \tag{SA.3.22}$$

Then using Lemma 7,

$$\lim_{B\to\infty} \limsup_{n\to\infty} \max_{(x',z')\in\mathcal{T}_{r'}(x,Z(x))} \mathbf{P}\left(\tilde{\mathfrak{X}}_{r_n}(x', z'; 1) > B/B' \,\middle|\, x', z'\right) \mathbf{1}\{|\mathcal{T}_{r'}(x, Z(x))| \leqslant B'\} \overset{a.s.}{=} 0.$$

Hence, by the bounded convergence theorem, for any $\varepsilon, B' > 0$, we can choose B large enough such that

$$\limsup_{n\to\infty} \text{(SA.3.21)} \leqslant \limsup_{n\to\infty} \text{(SA.3.22)} < \varepsilon/2.$$

Combined with (SA.3.20), this proves (SA.3.18).

Step 3. Steps 1 and 2 bounded the first probability on the right-hand side of (SA.3.15) below $\varepsilon/2$ for large n and B. Now consider the second probability on the right-hand side. Under the event $|J_0| \leqslant B$, note that $J_0 \subseteq \mathcal{N}_B(X, \mathcal{X}_n, \mathbf{W}, M(r_n))$. Hence,

$$\begin{aligned}
\mathbf{P}\left(\max_{j \in J_0} r_n^{-1}||X - X_j|| > C \cap |J_0| \leqslant B\right) &\leqslant \mathbf{P}\left(\max_{j \in \mathcal{N}_B(X, \mathcal{X}_n, \mathbf{W}, M(r_n))} r_n^{-1}||X - X_j|| > C\right) \\
&= \mathbf{P}\left(\max_{j \in \mathcal{N}_B(X, \tau_{X, r_n}\mathcal{X}_n, \mathbf{W}\mathbf{p}_{X, r_n}, M(1))} ||X - \tau_{X, r_n}X_j|| > C\right) \\
&\leqslant \mathbf{P}\left(\max_{(x', z') \in \mathbf{B}_{r_n}(X, Z(X); B)} ||X - x'|| > C\right) \\
&\leqslant \mathbf{P}\left(\max_{(x', z') \in \mathbf{B}_{\kappa}(X, Z(X); B)} ||X - x'|| > C\right),
\end{aligned}$$

where the second-to-last line follows from the proof of Lemma 9. By Lemma 8, the last line converges to zero as $C \to \infty$. Therefore, for any $\varepsilon > 0$ and $B > 0$, we can choose C large enough such that

$$\limsup_{n \to \infty} \mathbf{P}\left(\max_{j \in J_0} r_n^{-1}||X - X_j|| > C \cap |J_0| \leqslant B\right) < \varepsilon/2. \tag{SA.3.23}$$

Combining (SA.3.15), (SA.3.19), and (SA.3.23), we have

$$\lim_{C \to \infty} \limsup_{n \to \infty} \mathbf{P}\left(\max_{j \in J_0} r_n^{-1}||X - X_j|| > C\right) = 0.$$

■

SA.4 Proofs of Main Results

Proof of Proposition 1. Fix a realization of $\mathcal{X}_n$ and W, and let C^+ be a strategic neighborhood under this realization. Consider any $i, j \in C^+$ and $k \notin C^+$. Let $A \in \mathcal{A}(\mathcal{X}_n, W, r_n)$ and $A(-k) \in \mathcal{A}(\mathcal{X}_{\mathcal{N}_n \backslash \{k\}}, W_{\mathcal{N}_n \backslash \{k\}}, r_n)$. Since $k \notin C^+$, by construction, A_{ik} and A_{jk} are robustly absent, so by Assumption 1, these do not enter into S_{ij}. Then even if k is removed from the network, the current state of A_{ij} remains pairwise stable, or more formally, $A_{ij} = \mathbf{1}\{V(\delta_{ij}, S_{ij}(-k), W_{ij}) > 0\}$, where

$$S_{ij}(-k) = S(\delta_{ij}, \{\delta_{lm} : l, m \in \mathcal{N}_n \backslash \{k\}\}, W, A_{\mathcal{N}_n \backslash \{k\}, -ij}).$$

Since i, j, k are arbitrary, it follows that removing any node $k \notin C^+$ from the network has no impact on the pairwise stability of any A_{ij} for $i, j \in C^+$, as desired. ■

Proof of Theorem 1. The conditions of Theorem 2 are all satisfied by assumption, except Assumption 8. Part (a) of this assumption follows because we have assumed that $Z_i \perp\!\!\!\perp X_i$, i.e. H_z does not vary in its first argument. Part (b) of the latter assumption follows because under this independence assumption, $\mathbf{W}^*\mathbf{p}_{X,r} \stackrel{d}{=} \mathbf{W}^*\mathbf{p}_{\mathbf{0},1}$ for any $r \geqslant 0$, and $\{\mathbf{W}^*\mathbf{p}_{\mathbf{0},1}(y,y') : y,y' \in \mathcal{P}_{\kappa f(x)}\} \stackrel{d}{=} \{(Z(x'),Z(y'),\zeta(x',y')) : x',y' \in \mathcal{P}_{\kappa f(x)} \cup \{x\}\}$, the latter of which is the definition of W^∞ in §4. This implies

$$\int \mathbf{E}\left[\psi\left(x, \mathcal{P}_{\kappa f(x)} \cup \{x\}, \mathbf{W}^*\mathbf{p}_{x,r}, A(1)\right)\right] f(x)\,\mathrm{d}x \\ = \int \mathbf{E}\left[\psi\left(x, \mathcal{P}_{\kappa f(x)} \cup \{x\}, W^\infty, A(1)\right)\right] f(x)\,\mathrm{d}x$$

for any $r \geqslant 0$, in light of the functional notation introduced in §SA.1. This verifies part (b) and justifies the expression for the limit in Theorem 1. ∎

Proof of Theorem 2. We apply Theorem 3. First note that by definition of $S(\cdot)$ and $\lambda(\cdot)$, positions only directly enter the model through distances δ_{ij} (they may enter "indirectly" through $\mathbf{W}$). Also, the dissimilarity measures may be rewritten as

$$\{\delta_{ij} : i,j \in \mathcal{N}_n\} \equiv \{r_n^{-1}||X_i - X_j|| : i,j \in \mathcal{N}_n\} = \{||X' - X''|| : X', X'' \in \tau_{X,r_n}\mathcal{X}_n\}.$$

Furthermore, $\mathbf{W}\mathbf{p}_{X,r_n}(\tau_{X,r_n}\mathcal{X}_n) = \mathbf{W}(\mathcal{X}_n)$ by definition. Consequently, the following two finite models are equivalent for any $X \in \mathcal{X}_n$:

$$(V, \lambda, \mathcal{X}_n, \mathbf{W}, r_n) \quad \text{and} \quad (V, \lambda, \tau_{X,r_n}\mathcal{X}_n, \mathbf{W}\mathbf{p}_{X,r_n}, 1).$$

Therefore, (14) can be restated as

$$\sup_{n\in\mathbb{N}} \mathbf{E}\left[||\psi\big(X_i, \tau_{X_i,r_n}\mathcal{X}_n, \mathbf{W}\mathbf{p}_{X_i,r_n}, A(1)\big)||^{2+\epsilon}\right] < \infty, \tag{SA.4.1}$$

and network moments can be rewritten as

$$\frac{1}{n}\sum_{i=1}^n \psi\big(X_i, \tau_{X_i,r_n}\mathcal{X}_n, \mathbf{W}\mathbf{p}_{X_i,r_n}, A(1)\big).$$

Since A is a deterministic functional of positions, attributes, and the sparsity parameter by Assumption 2, we can define

$$\xi(X_i, \tau_{X_i,r_n}\mathcal{X}_n, \mathbf{W}\mathbf{p}_{X_i,r_n}) \equiv \psi(X_i, \tau_{X_i,r_n}\mathcal{X}_n, \mathbf{W}\mathbf{p}_{X_i,r_n}, A(1)),$$

putting us in the setup of Theorem 3. By (SA.4.1), (SA.2.3) holds. By Assumption 8, $\mathbf{E}[\xi(x, \mathcal{P}_{\kappa f(x)}, \mathbf{Wp}_{x,r})]$ is continuous at $r = 0$. Thus, it remains to verify stabilization (Definition 4).

Let $J_0 \equiv J(X, \tau_{X,r_n}\mathcal{X}_n, \mathbf{Wp}_{X,r_n}, 1)$ be the relevant set (18) of the node positioned at X under the finite model

$$(V, \lambda, \tau_{X,r_n}\mathcal{X}_n, \mathbf{Wp}_{X,r_n}, 1).$$

Note this is equivalent to the definition of $J(X, \mathcal{X}_n, \mathbf{W}, r_n)$ prior to Lemma 9 in §SA.3. By Lemma 1 in §5, $\xi(X, \tau_{X,r_n}\mathcal{X}_n, \mathbf{Wp}_{X,r_n})$ only depends on its arguments through the positions and attributes of nodes in J_0 under the finite model. An identical argument holds under the limit model

$$(V, \lambda, \mathcal{P}_{\kappa f(X)} \cup \{X\}, \mathbf{Wp}_{X,r_n}, 1).$$

Take $\mathbf{R}_n$ in Definition 4 to be equal to $\mathbf{R}_n(X)$ in equation (SA.3.5) of §SA.3. Then part (b) of Definition 4 follows from Proposition 1 and part (a) from Theorem 4 in §SA.3. ∎

Proof of Proposition 2. We apply Theorem 1. First we write subgraph counts as averages of K-local node statistics in the sense of Assumption 5. Let $A = \{i_1, \dots, i_K\} \subseteq \mathcal{N}_n$. For $g \in \mathcal{G}_K$, define

$$\psi(X_{i_1}, \mathcal{X}_n, W, A(r_n)) = \sum_{i_2,\dots,i_K=1}^{n} \mathbf{1}\{\Gamma(X_{i_1}, \dots, X_{i_K}; \mathcal{X}_n, r_n) = g\},$$

so that the left-hand side of (21) equals $n^{-1}\sum_{i=1}^{n} \psi(X_i, \mathcal{X}_n, W, A(r_n))$. Because by definition $\mathbf{1}\{\Gamma(X_{i_1}, \dots, X_{i_K}; \mathcal{X}_n, r_n) = g\}$ is nonzero only when $\Gamma(X_{i_1}, \dots, X_{i_K}; \mathcal{X}_n, r_n)$ is a connected subnetwork, it follows that ψ satisfies Assumption 5. For example if $K = 2$, so that the network moment counts connected pairs, then ψ is the degree of node i, which is 1-local.

It remains to verify (SA.4.1). Since g is a connected subnetwork, the number of subsets $\{i_1, \dots, i_K\}$ for which the event $\{\Gamma(X_{i_1}, \dots, X_{i_K}; \mathcal{X}_n, r_n) = g\}$ occurs is bounded above by the number of such subsets that are *at least* minimally connected in A, which corresponds to the event

$$\{A_{i_1,i_2} \cdot \dots \cdot A_{i_{K-1},i_K} = 1\}. \tag{SA.4.2}$$

For example if $K = 3$, then s is either an intransitive triad, i.e. $A_{ij}A_{jk}(1 - A_{ik}) = 1$, or a transitive triad, i.e. $A_{ij}A_{jk}A_{ik} = 1$. Both are bounded above by $A_{ij}A_{jk}$. Furthermore, (SA.4.2) occurs only if

$$\{M_{i_1,i_2}(r_n) \cdot \dots \cdot M_{i_{K-1},i_K}(r_n) = 1\}$$

occurs, since $A_{ij} \leqslant M_{ij}(r_n)$. Therefore, there exists a constant C such that

$$\mathbf{E}\left[\psi(X_{i_1}, \mathcal{X}_n, W, A(r_n))^{2+\epsilon}\right] \leqslant C\mathbf{E}\left[\left(\sum_{i_2=1}^{n} \cdots \sum_{i_K=1}^{n} M_{i_1,i_2}(r_n) \cdot \ldots \cdot M_{i_{K-1},i_K}(r_n)\right)^{2+\epsilon}\right].$$

Then (SA.4.1) follows from (20).

Lastly, we obtain the expression for the limit in (21) by first applying the law of iterated expectations (conditioning on $\mathcal{P}_{\kappa f(x_1)}$) and second the Slivnyak-Mecke formula (Schneider and W., 2008, Corollary 3.2.3). ■

SA.5 Primitive Sufficient Conditions

SA.5.1 Uniform Square-Integrability

We first discuss some examples for which equation (22) holds.

Example 9. *Suppose $A_{ij} \leqslant \mathbf{1}\{\delta_{ij} \leqslant 1\}$, as in Example 6. Then $M(r_n)$ is a subnetwork of the random geometric graph where nodes i and j are connected if and only if $\delta_{ij} \leqslant 1$. The conditional expected degree of a node in this graph is asymptotically bounded uniformly over node types using Lemma 2 in §B.*

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Example 10. *Consider the model in Example 1 with $h(Z_i, Z_j; \theta_1) = \theta_1$ and $\rho(\delta_{ij}) = -\delta_{ij}$. Suppose the random shock has polynomial tails: for sufficiently large z that $\mathbf{P}(\zeta_{ij} > z) \leqslant cz^{-d}$ for some constant c and $z \geqslant 0$, where d is the dimension of X_i. Then for n sufficiently large,*

$$n\mathbf{E}\left[M_{ij}(r_n) \,|\, X_i, Z_i\right] \leqslant c\,\mathbf{E}\left[n\left(-\delta_{ij} - \theta_1 - \bar{S}'\theta_2\right)^{-d} \,|\, X_i\right].$$

Then (22) follows from the definition of r_n.

Next, we discuss how (22) implies assumption (20) in Proposition 2 for $\epsilon = 1$. Specifically, we consider the case $K = 2$ and discuss how the argument generalizes. To simplify notation we suppress the dependence of $M_{ij}(r_n)$ on r_n. We need to show that

$$\sup_n \mathbf{E}\left[\left(\sum_{j=1}^n \sum_{k=1}^n M_{ik}M_{kj}\right)^3\right] < \infty.$$

By the AM-GM inequality, the term in the expectation is bounded above by

$$\sum_{j,k,l} M_{ij}M_{ik}M_{il}\frac{1}{3}\left[\left(\sum_a M_{ja}\right)^3 + \left(\sum_b M_{kb}\right)^3 + \left(\sum_c M_{lc}\right)^3\right].$$

The others being similar, consider the term

$$\frac{1}{3}\sum_{j,k,l}\sum_{a_1,a_2,a_3} M_{ij}M_{ik}M_{il}M_{ja_1}M_{ja_2}M_{ja_3}. \tag{SA.5.1}$$

Consider the part of the sum where all the indices are different ($i \neq j \neq k \neq l \neq a_1 \neq a_2 \neq a_3$). This equals $(3n)^{-1}$ times a count of the number of septuplets whose subnetwork in M is at least minimally connected. By "minimally connected" we mean the removal of any link results in a disconnected subnetwork. For parts of the sum where some indices coincide, this may result in a count for a subnetwork that is not minimally connected, but such a count can always be bounded above by an appropriate count for a minimally connected subnetwork. For example, if $a_1 = k$ but all other indices differ, then the corresponding term of (SA.5.1) is given by

$$\frac{1}{3}\sum_{j,k,l}\sum_{a_2,a_3} M_{ij}M_{ik}M_{il}M_{jk}M_{ja_2}M_{ja_3} \leqslant \frac{1}{3}\sum_{j,k,l}\sum_{a_2,a_3} M_{ij}M_{ik}M_{il}M_{ja_2}M_{ja_3}.$$

The left-hand side counts the occurrence of a subnetwork that is not minimally connected (it has a cycle between i, j, k), unlike the right-hand side.

Thus, consider the part of the sum (SA.5.1) where all the indices are different, as the argument for the other terms is the same once we bound them above by an appropriate minimally-connected-subgraph count. Then the expectation of the summand equals

$$\begin{aligned}\mathbf{E}\left[M_{ij}M_{ik}M_{il}M_{ja_1}M_{ja_2}M_{ja_3}\right] \leqslant \mathbf{E}[M_{ij}] \sup_{X_k,Z_k} \mathbf{E}[M_{ik}\,|\,X_k,Z_k] \sup_{X_l,Z_l} \mathbf{E}[M_{il}\,|\,X_l,Z_l] \\ \sup_{X_{a_1},Z_{a_1}} \mathbf{E}[M_{ja_1}\,|\,X_{a_1},Z_{a_1}] \sup_{X_{a_2},Z_{a_2}} \mathbf{E}[M_{ja_2}\,|\,X_{a_2},Z_{a_2}] \sup_{X_{a_3},Z_{a_3}} \mathbf{E}[M_{ja_3}\,|\,X_{a_3},Z_{a_3}] \\ \leqslant \left(\sup_{X_i,Z_i} \mathbf{E}[M_{ij}\,|\,X_i,Z_i]\right)^6.\end{aligned}$$

The first inequality above is attained by iteratively conditioning on and taking the supremum over attributes of leaf nodes (nodes with only one connection), i.e. nodes k, l, a_1, a_2, a_3, and then using the assumption that $\{(X_i, Z_i)\}_{i=1}^n$ is i.i.d.

The previous derivation yields

$$\sup_n \frac{1}{3} \sum_{j \neq k \neq l \neq a_1 \neq a_2 \neq a_3} \mathbf{E}\left[M_{ij} M_{ik} M_{il} M_{ja_1} M_{ja_2} M_{ja_3}\right] \leqslant \sup_n \frac{1}{3} \left(n \sup_{X_i, Z_i} \mathbf{E}[M_{ij} \mid X_i, Z_i]\right)^6,$$

which is finite by Assumption 22. This proves the claim for $K = 2$.

For general K, we can follow the same line of reasoning, bounding

$$\left(\sum_{i_2=1}^n \cdots \sum_{i_K=1}^n M_{i_1,i_2}(r_n) \cdot \cdots \cdot M_{i_{K-1},i_K}(r_n)\right)^3$$

from above by the sum of K-tuplets that form at least minimally connected subnetworks and then repeatedly applying (22).

SA.5.2 Continuity

This subsection discusses conditions for Assumption 8(b). This simplest case is when positions are independent of attributes; this is covered in Theorem 1. Another simple case is the average degree, where $\psi_i = \sum_j A_{ij}$, and there are no strategic interactions. Then $A_{ij} = M_{ij}(r_n)$, and Assumption 8(b) follows from Assumption 8(a) by the proof of Lemma 2 in §B. We point out this case because the lemma provides a simple illustration of the relevance of the continuity assumption.

The remainder of this subsection focuses on the case of subgraph counts, allowing for strategic interactions and dependence between positions and attributes. The main conditions required are continuity of V in its arguments, continuity of the distribution of ζ_{ij}, and Assumption 8(a), which are mild regularity conditions. The method of proof appears to be more broadly applicable to other moments.

Fix $K \in \mathbb{N}$, and let $\tilde{\mathcal{G}}_K$ be the set of all possible (labeled) subgraphs on K nodes. Following the notation in §6.1, define the *subgraph distribution*

$$P_K(G; r_n) = \mathbf{P}\left(\Gamma(x_1, \dots, x_K; \mathcal{P}_{\kappa f(x_1)} \cup \{x_1, \dots, x_K\}, \mathbf{W}\mathbf{p}_{x_1, r_n}, 1) \in G\right),$$

where $G \subseteq \tilde{\mathcal{G}}_K$. We seek conditions under which there exists a selection mechanism such that

$$P_K(G; r) \text{ is continuous at } r = 0. \tag{SA.5.2}$$

Then, *assuming this selection mechanism rationalizes the data*, Assumption 8(b) follows from (20) and the dominated convergence theorem, since subgraph moments correspond to the case $G = \{g\}$ for some connected subgraph g. The remainder of this subsection is concerned with deriving primitive conditions under which such a selection mechanism exists.

We first introduce some notation. Let $\mathcal{E}_K(\mathcal{P}_{\kappa f(x_1)}, \mathbf{W}\mathbf{p}_{x_1,r_n})$ be the set of subgraphs on K nodes that can be supported in a pairwise stable network under the limit model

$$(V, \lambda, \mathcal{P}_{\kappa f(x_1)} \cup \{x_1, \dots, x_K\}, \mathbf{W}\mathbf{p}_{x_1,r_n}, 1). \tag{SA.5.3}$$

By "supported in a pairwise stable network," we mean that this is the set of subgraphs $g \in \tilde{\mathcal{G}}_K$ for which there exists a network A on $\mathcal{P}_{\kappa f(x_1)} \cup \{x_1, \dots, x_K\}$ such that A is pairwise stable and the subgraph of A on nodes $x_1, \dots, x_K$ coincides with g. Let $\Delta(S)$ denote the set of probability distributions supported on a set S. Then $\Delta(\mathcal{E}_K(\mathcal{P}_{\kappa f(x_1)}, \mathbf{W}\mathbf{p}_{x_1,r_n}))$ is the set of *subgraph selection mechanisms* on $\tilde{\mathcal{G}}_K$.[1]

The subgraph distribution $P_K(\,\cdot\,; r_n)$ is an element of the set

$$\mathcal{P}_K(r_n) = \Big\{ P'_K(\,\cdot\,; r_n) \in \Delta(\tilde{\mathcal{G}}_K) : \exists\, \sigma_K(\,\cdot\,; \mathcal{P}_{\kappa f(x_1)}, \mathbf{W}\mathbf{p}_{x_1,r_n}) \in \Delta\left(\mathcal{E}_K(\mathcal{P}_{\kappa f(x_1)}, \mathbf{W}\mathbf{p}_{x_1,r_n})\right) \\ \text{such that } P'_K(G; r_n) = \mathbf{E}[\sigma_K(G; \mathcal{P}_{\kappa f(x_1)}, \mathbf{W}\mathbf{p}_{x_1,r_n})]\ \forall G \subseteq \tilde{\mathcal{G}}_K \Big\}.$$

This is the set of distributions on $\tilde{\mathcal{G}}_K$ that can be generated by some subgraph selection mechanism, and therefore, implicitly, some selection mechanism λ. Without strategic interactions, this set is a singleton, since the model is complete. Then (SA.5.2) follows more or less immediately from Assumption 8(a). With strategic interactions, the challenge is that there need not be any relationship between the subgraph count distribution selected from $\mathcal{P}_K(r)$ and that selected from $\mathcal{P}_K(r')$, which is problematic for (SA.5.2). We next derive primitive conditions under which these selections can be "chosen continuously" as r varies, so that (SA.5.2) holds for some selection mechanism.

Let the *limit capacity* $\mathcal{L}(\,\cdot\,; r_n) : G \subseteq \tilde{\mathcal{G}}_K \mapsto [0, 1]$ be the set function satisfying

$$\mathcal{L}(G; r_n) = \mathbf{P}\left(\mathcal{E}_K(\mathcal{P}_{\kappa f(x_1)}, \mathbf{W}\mathbf{p}_{x_1,r_n}) \cap G \neq \varnothing\right).$$

That is, $\mathcal{L}(G; r_n)$ is the probability that all subgraphs in G can be supported in a pairwise

[1] This usage of "selection mechanism" is in accordance with more the standard notion of a conditional distribution over the set of equilibria. It differs from the definition of λ in Assumption 2.

stable network under the limit model. We can define the analog of the limit capacity $\mathcal{L}(G;r)$ under a finite model. Let $\mathcal{E}_K(\tau_{x_1,r_n}\mathcal{X}_n, \mathbf{Wp}_{x_1,r})$ be the set of subgraphs on K nodes that can be supported in a pairwise stable network under the finite model

$$(V, \lambda, \tau_{x_1,r_n}\mathcal{X}_n \cup \{x_1, \dots, x_K\}, \mathbf{Wp}_{x_1,r}, 1). \tag{SA.5.4}$$

Define the *finite capacity*

$$\mathcal{L}_n(G;r) = \mathbf{P}\left(\mathcal{E}_K(\tau_{x_1,r_n}\mathcal{X}_n, \mathbf{Wp}_{x_1,r}) \cap G \neq \varnothing\right).$$

Assumption 9 (Continuous Finite Capacity)**.** *For any n, $\mathcal{L}_n(G;r)$ is continuous in $r \in [0,\kappa]$, for all $G \in \tilde{\mathcal{G}}_K$.*

This assumption is easy to verify. For instance, if Assumption 8(a) holds, ζ_{ij} is continuously distributed, V is continuous in (Z_i, Z_j), and S_{ij} does not depend on attributes as in Example 1, then Assumption 9 holds. More generally, if S_{ij} depends on attributes, then it needs to be continuous in them.

Theorem 5. *Under Assumptions 1, 4, 6, and 9, there exists a continuous selection of the correspondence $\mathcal{P}_K(r)$. That is, there exists $P_K(\,\cdot\,;\,\cdot\,)$ such that for all $r \in [0,\kappa]$, we have $P_K(\,\cdot\,;r) \in \mathcal{P}_K(r)$, and $P_K(G;r)$ is continuous in $r \in [0,\kappa]$ for all $G \in \tilde{\mathcal{G}}_K$.*

Thus, under the stated assumptions, there exists a selection mechanism such that (SA.5.2) holds, which justifies Assumption 8(b) for the case of subgraph counts. The proof proceeds in two steps, given by the two lemmas below. First, we show that the conclusion of the theorem holds if the limit capacity is continuous. Then we show that continuity of the limit capacity follows from the assumptions of the theorem.

Assumption 10 (Continuous Limit Capacity)**.** *$\mathcal{L}(G;r)$ is continuous in $r \in [0,\kappa]$, for all $G \in \tilde{\mathcal{G}}_K$.*

Lemma 10. *Under Assumption 10, there exists a continuous selection of the correspondence $\mathcal{P}_K(r)$.*

The formal proof can be found below. Here is a brief sketch. The set $\mathcal{P}_K(r)$ can be characterized as the core of the Choquet capacity $\mathcal{L}(\,\cdot\,;r)$ (see Definition 1 of Galichon and Henry, 2011). If this capacity is continuous in r, we can show that the correspondence $\mathcal{P}_K(r)$ is lower hemicontinuous in r. Then by Michael's selection theorem, there exists a continuous selection $P_K(\cdot\,;r)$.

Lemma 11. *Assumptions 1, 4, 6, 8(a), and 9 imply Assumption 10.*

To prove this lemma, we show that the finite capacity converges to the limit capacity uniformly in r. This is shown using Proposition 1 in §3.1, which states that $\mathcal{E}_K$ only depends on its arguments through the strategic neighborhoods of nodes positions at $x_1, \dots, x_K$. A branching process argument (see §SA.3) establishes that these neighborhoods are asymptotically bounded uniformly in r. Convergence then follows from an argument similar to the proof of Lemma 4.

Proof of Lemma 10. Notice that $\mathcal{P}_K(r)$ is closed, non-empty, and convex for any $r \in [0, \kappa]$. Then if $\mathcal{P}_K(r)$ is lower hemicontinuous in r, the result follows from Michael's selection theorem (Michael, 1956). Hence, it suffices to show lower hemicontinuity.

Notice that $\mathcal{L}(\cdot\,;r)$ is a Choquet capacity. Following the proof of Theorem 1 of Galichon and Henry (2011), $\mathcal{P}_K(r)$ is equivalent to the core of this capacity, i.e.

$$\mathcal{P}_K(r) = \left\{ P'_K(\cdot\,;r) \in \Delta(\tilde{\mathcal{G}}_K) : P'_K(G;r) \leqslant \mathcal{L}(G;r)\ \forall G \in \tilde{\mathcal{G}}_K \right\}. \tag{SA.5.5}$$

We next show that continuity of the capacity implies lower hemicontinuity of the core. Let $\{r'_n\}$ be any sequence in $[0, \kappa]$ converging to r, and let $\tilde{p}'_K(\cdot\,;r) \in \mathcal{P}_K(r)$. Consider an arbitrary subsequence $\{r_{m(n)}\}$ of $\{r_n\}$. Define

$$\tilde{p}'_K(\cdot\,;r_{m(n)}) = \mathcal{L}(\cdot\,;r_{m(n)}) - (\mathcal{L}(\cdot\,;r) - p'_K(\cdot\,;r)).$$

We claim that $\tilde{p}'_K(\cdot\,;r_{m(n)}) \in \mathcal{P}_K(r_{m(n)})$. First, since $p'_K(\cdot\,;r) \in \mathcal{P}_K(r)$, by (SA.5.5), $p'_K(\cdot\,;r) \leqslant \mathcal{L}(\cdot\,;r)$, so $\tilde{p}'_K(\cdot\,;r_{m(n)}) \leqslant \mathcal{L}(\cdot\,;r_{m(n)})$. Second, $\sum_{g \in \tilde{\mathcal{G}}_K} \tilde{p}'_K(g;r_{m(n)}) = 1-(1-1) = 1$. Third, $\tilde{p}'_K(\cdot\,;r_{m(n)})$ is clearly supported on $\tilde{\mathcal{G}}_K$. This establishes the claim.

It remains to show that $\tilde{p}'_K(\cdot\,;r_{m(n)}) \to p'_K(\cdot\,,r)$ uniformly in $\tilde{\mathcal{G}}_K$. This follows since continuity of the capacity implies $\mathcal{L}(\cdot\,;r_{m(n)}) \to \mathcal{L}(\cdot\,;r)$ uniformly in $\tilde{\mathcal{G}}_K$. ∎

Proof of Lemma 11. Under Assumption 9, the finite capacity is continuous in $r \in [0, \kappa]$. To show that the limit capacity inherits this continuity property, we prove that the finite capacity converges to the limit capacity as $n \to \infty$, uniformly over values of r in some neighborhood of zero.

By Proposition 1, $\mathcal{E}_K$ under both the finite and limit models only depends on its arguments through the positions and attributes of nodes in $\cup_{i=1}^K C_i^+$, the strategic neighborhoods of nodes positioned at $x_1, \dots, x_K$. To make this statement formal, we need several pieces of notation. Let us abbreviate $\mathcal{E}_K(\tau_{x_1,r_n}\mathcal{X}_n, r) \equiv \mathcal{E}_K(\tau_{x_1,r_n}\mathcal{X}_n, \mathbf{Wp}_{x_1,r})$ and $\mathcal{E}_K(\mathcal{P}_{\kappa f(x_1)}, r) = \mathcal{E}_K(\mathcal{P}_{\kappa f(x_1)}, \mathbf{Wp}_{x_1,r})$. Let $\mathcal{X}$ generically denote the set of node positions under either the finite (SA.5.4) or limit (SA.5.3) model. Let $C^+(x, \mathcal{X}, r)$ denote the set of positions corresponding to nodes in the strategic neighborhood of a node positioned at $x \in \mathcal{X}$ under the finite/limit model with attributes given by $\mathbf{Wp}_{x_1,r}(\mathcal{X})$, and $C_i^+ = C^+(x_i, \mathcal{X}, r)$. Define

$$\mathbf{R}(\mathcal{X}, r) = \max_{X' \in \bigcup_{i=1}^K C^+(x_i,\mathcal{X},r)} ||x_1 - X'||.$$

Finally, let $\tau_{x_1,r_n}\mathcal{X}_n|_{C^+} = \tau_{x_1,r_n}\mathcal{X}_n \cap B(x_1, \mathbf{R}(\tau_{x_1,r_n}\mathcal{X}_n, r))$ and $\mathcal{P}_{\kappa f(x_1)}|_{C^+} = \mathcal{P}_{\kappa f(x_1)} \cap B(x_1, \mathbf{R}(\mathcal{P}_{\kappa f(x_1)}, r))$. These are the subset of positions lying within the smallest neighborhood centered at x_1 that contains the positions of all nodes in $\cup_{i=1}^K C_i^+$.

We had just argued that Proposition 1 implies

$$\mathcal{E}_K(\tau_{x_1,r_n}\mathcal{X}_n, r) = \mathcal{E}_K(\tau_{x_1,r_n}\mathcal{X}_n|_{C^+}, r) \quad \text{and} \quad \mathcal{E}_K(\mathcal{P}_{\kappa f(x_1)}, r) = \mathcal{E}_K(\mathcal{P}_{\kappa f(x_1)}|_{C^+}, r). \quad \text{(SA.5.6)}$$

Then mimicking the argument for (SA.2.6), for some ρ specified later,

$$\mathbf{P}\left(\sup_{r \in [0,\rho]} \left|\mathbf{1}\left\{\mathcal{E}_K(\tau_{x_1,r_n}\mathcal{X}_n, r) \cap G \neq \varnothing\right\} - \mathbf{1}\left\{\mathcal{E}_K(\mathcal{P}_{\kappa f(x_1)}, r) \cap G \neq \varnothing\right\}\right| > \epsilon\right)$$
$$\leqslant \mathbf{P}(\mathcal{E}_X^n(R)^c) + \mathbf{P}(\sup_{r \in [0,\rho]} \mathbf{R}_n(r) > R), \quad \text{(SA.5.7)}$$

where $E_X^n(R)$ is defined prior to (SA.2.6) and

$$\mathbf{R}_n(r) \equiv \max\{\mathbf{R}(\tau_{x_1,r_n}\mathcal{X}_n, r), \mathbf{R}(\mathcal{P}_{\kappa f(x_1)}, r)\}.$$

By Lemma 3, for any $R > 0$, $\lim_{n\to\infty} \mathbf{P}(E_X^n(R)^c) = 0$. Suppose for the moment that

$$\sup_{r \in [0,\rho]} \mathbf{R}_n(r) = O_p(1). \quad \text{(SA.5.8)}$$

Then we would have $\lim_{n\to\infty} \lim_{R\to\infty} \mathbf{P}(\sup_{r\in[0,\rho]} \mathbf{R}_n(r) > R) = 0$, thereby establishing that the limit of the left-hand side of (SA.5.7) is zero. Furthermore, by the bounded convergence theorem,

$$0 = \lim_{n\to\infty} \mathbf{E}\left[\sup_{r\in[0,\rho]} \left|\mathbf{1}\left\{\mathcal{E}_K(\tau_{x_1,r_n}\mathcal{X}_n, r)\cap G \neq \varnothing\right\} - \mathbf{1}\left\{\mathcal{E}_K(\mathcal{P}_{\kappa f(x_1)}, r)\cap G \neq \varnothing\right\}\right|\right]$$
$$\geqslant \lim_{n\to\infty} \sup_{r\in[0,\rho]} \left|\mathbf{P}\left(\mathcal{E}_K(\tau_{x_1,r_n}\mathcal{X}_n, r)\cap G \neq \varnothing\right) - \mathbf{P}\left(\mathcal{E}_K(\mathcal{P}_{\kappa f(x_1)}, r)\cap G \neq \varnothing\right)\right|$$
$$= \lim_{n\to\infty} \sup_{r\in[0,\rho]} \big|\mathcal{L}_n(G;r) - \mathcal{L}(G;r)\big|,$$

which would complete the proof. Therefore, it suffices to show (SA.5.8). By an argument similar to the proof of Theorem 4, (SA.5.8) holds if

$$\sup_{r\in[0,\rho]} \big| \cup_{i=1}^K C^+(x_i, \mathcal{X}, r)\big| = O_p(1).$$

We will prove this for the finite model, as the argument for the limit model is similar.

Recall that we are considering the network-formation model

$$(V, \lambda, \tau_{x_1,r_n}\mathcal{X}_n \cup \{x_1,\dots,x_K\}, \mathbf{W}\mathbf{p}_{x_1,r}, 1)$$

(cf. Assumption 3). Also, recall that $C^+(x_i, \tau_{x_1,r_n}\mathcal{X}_n \cup \{x_1,\dots,x_K\}, r)$ denotes the strategic neighborhood of node positioned at x_i under this model. Observe that

$$\cup_{i=1}^K C^+(x_i, \tau_{x_1,r_n}\mathcal{X}_n \cup \{x_1,\dots,x_K\}, r) \subseteq \cup_{i=1}^K C^+(x_i, \tau_{x_1,r_n}\mathcal{X}_n \cup \{x_i\}, r),$$

where $C^+(x_i, \tau_{x_1,r_n}\mathcal{X}_n \cup \{x_i\}, r)$ is the strategic neighborhood in the aforementioned finite model but with nodes positioned at $x_2,\dots,x_K$ removed from the game. It is therefore enough to show that

$$\sup_{r\in[0,\rho]} \big|C^+(x_i, \tau_{x_1,r_n}\mathcal{X}_n \cup \{x_i\}, r)\big| = O_p(1) \tag{SA.5.9}$$

for any $i = 1,\dots,K$. This follows from a branching process argument that is the subject of §SA.3. In particular, by Lemma 9, the strategic neighborhood size is stochastically dominated by $\hat{\mathfrak{X}}_r(x_i, Z(x_i); K)$, which in turn is stochastically dominated by $\hat{\mathfrak{X}}_\kappa(x_i, Z(x_i); K)$, which does not depend on r. The latter is a.s. finite by step 1 of the proof of Theorem 4. ■

Proof of Theorem 5. By Lemma 11, Assumption 10 holds. The result follows from Lemma 10. ■

SA.6 Simulating Limit Expectations

This section gives details on how we simulate the limit expectation of Theorem 1 in §7. First observe that

$$\psi\left(x, \mathcal{P}_{\kappa f(x)} \cup \{x\}, W^\infty, A(1)\right) \stackrel{d}{=} \psi\left(\mathbf{0}, \mathcal{P}_{\kappa f(x)} \cup \{\mathbf{0}\}, W^\infty, A(1)\right),$$

where $\mathbf{0}$ is the origin of $\mathbb{R}^d$, since positions only enter the model through differences $||X_i - X_j||$, $\mathcal{P}_\kappa \stackrel{d}{=} \mathcal{P}_\kappa - x$, and attributes are i.i.d. conditional on positions. Since f is uniform, the right-hand side of (15) then simplifies to

$$\mathbf{E}\left[\psi\left(\mathbf{0}, \mathcal{P}_\kappa \cup \{\mathbf{0}\}, W^\infty, A(1)\right)\right].$$

This can be computed by repeatedly simulating the limit model, computing the node statistic for the node at $\mathbf{0}$, and then averaging across simulations. It is impossible to simulate $\mathcal{P}_\kappa$, since it is countably infinite. Instead, we simulate its restriction to $[-10, 10]^2$, which should suffice for several reasons. First, we only need to compute the node statistic for the node at the origin. Second, nodes only link with those at most distance one away in this example. Third, strategic neighborhood sizes are bounded in probability (and typically quite small), so a large enough grid should be very likely to contain this neighborhood.

To simulate from the process restricted to $[-10, 10]^2$, we make use of the fact that the following process has the same distribution: $\{X_1, \dots, X_{N_{400\kappa}}\}$, where $\{X_i\} \stackrel{iid}{\sim} U([0,1]^2)$ is independent of $N_{400\kappa} \sim \text{Poisson}(400\kappa)$ (see e.g. Penrose, 2003, Ch. 1.7). This is straightforward to simulate. We can then simulate attributes and $A(1)$ in the same way as the finite model, except that we set $r_n = 1$.

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